

# GLOBAL WELL-POSEDNESS AND SCATTERING FOR SMALL DATA FOR THE 3-D KADOMTSEV-PETVIASHVILI-II EQUATION

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**ABSTRACT.** We study global well-posedness for the Kadomtsev-Petviashvili II equation in three space dimensions with small initial data. The crucial points are new bilinear estimates and the definition of the function spaces. As by-product we obtain that all solutions to small initial data scatter as  $t \rightarrow \pm\infty$ .

**Keywords:** Kadomtsev-Petviashvili II, Galilean transform, Bilinear estimate, nonlinear waves.

## 1. INTRODUCTION AND MAIN RESULTS

In this paper, we study the Cauchy problem for the 3-dimensional Kadomtsev-Petviashvili II (KP-II) equation

$$(1.1) \quad \begin{cases} \partial_x (\partial_t u + \partial_x^3 u + \partial_x(u^2)) + \Delta_y u = 0 & (t, x, y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \\ u(0, x, y) = u_0(x, y) & (x, y) \in \mathbb{R} \times \mathbb{R}^2. \end{cases}$$

The Kadomtsev-Petviashvili (KP) equations describe nonlinear wave interactions of almost parallel waves. They come with at least four different flavors: The KP-II equation for which the line soliton is supposed to be stable, the KP-I equation with localized solitons, and the modified KP-I and KP-II equations with cubic nonlinearities.

The KP-II equation is invariant under

- i) Translations in  $x, y$  and  $t$ .
- ii) Scaling:  $\lambda^2 u(\lambda x, \lambda^2 y, \lambda^3 t)$  is a solution if  $u$  satisfies the KP-II equation (1.1).
- iii) Galilean transform: Let  $c \in \mathbb{R}^2$ . Then  $u(t, x - c \cdot y - |c|^2 t, y + 2ct)$  is a solution if  $u$  satisfies (1.1). On the Fourier side the transform is  $\hat{u}(\tau - |c|^2 \xi - 2c \cdot \eta, \xi, \eta + c\xi)$  where  $\tau$  is the Fourier variable of  $t$ ,  $\xi \in \mathbb{R}$  is the Fourier variable of  $x$  and  $\eta$  the one of  $y$ .
- iv) Isometries of the  $y$  plane.
- v) Simultaneous reflections of  $x, t$  and  $u$ .

The Galilean invariance is often a consequence of the rotational symmetry of full systems for which certain solutions are asymptotically described by a KP equation. The interest in the KP equations comes from the expectation that they describe waves in a certain asymptotic regime for a large class of problems, for which one

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does not even have to formulate a full model, similar to the role of the nonlinear Schrödinger equation in nonlinear optics.

The Galilean symmetry group is noncompact, in contrast to the orthogonal group  $O(n)$  and it seems that with this noncompactness the difficulty increases with the dimension, in contrast to what is true for many wave and Schrödinger equations. It would be interesting to see whether the stronger decay of the linear equation compared to the 2d problem can be used to prove global existence for small Schwartz functions.

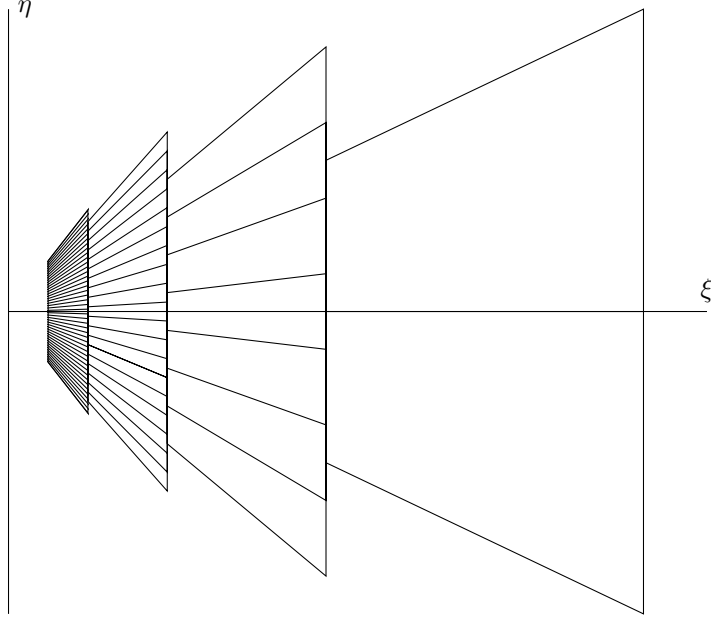
We search for spaces of initial data and solutions which reflect the symmetries. Given  $\lambda \in \mathbb{R} \setminus \{0\}$ , we define the Fourier projection  $u_\lambda$  (we denote the Fourier transform by  $\mathcal{F}$  resp.  $\gamma$ ) by

$$(1.2) \quad \hat{u}_\lambda(\tau, \xi, \eta) = \begin{cases} \hat{u}(\tau, \xi, \eta) & \text{if } \lambda \leq |\xi| < 2\lambda \\ 0 & \text{otherwise.} \end{cases}$$

We will always choose  $\lambda$  to be a power of 2. For fixed  $\lambda$ , we partition the set  $\{(\xi, \eta) \in \mathbb{R} \times \mathbb{R}^2 : \lambda \leq |\xi| < 2\lambda\}$  into sets  $\Gamma_{\lambda,k}$  for  $k \in \lambda \cdot \mathbb{Z}^2$  defined by

$$(1.3) \quad \Gamma_{\lambda,k} = \left\{ (\xi, \eta) : \lambda \leq |\xi| < 2\lambda, \left| \frac{\eta}{\xi} - k \right|_\infty \leq \frac{\lambda}{2} \right\}$$

where  $|a|_\infty = \max\{|a_1|, |a_2|\}$ . This decomposition is shown below.



For  $1 \leq q < \infty$ ,  $1 \leq p < \infty$ , a tempered distribution  $f$  is said to be in  $l^q l^p L^2$  if it is in the closure of  $C_0^\infty$  with respect to the norm

$$\|f\|_{l^q l^p L^2} := \left\{ \sum_{\lambda \in 2^{\mathbb{Z}}} \lambda^{\frac{q}{2}} \left( \sum_{k \in \lambda \cdot \mathbb{Z}^2} \|f_{\Gamma_{\lambda,k}}\|_{L^2}^{\frac{q}{p}} \right)^{\frac{1}{q}} \right\} < \infty.$$

The case  $p, q = \infty$  require the standard modification. Here and in the sequel  $f_{\Gamma_{\lambda,k}}$  denotes the Fourier projection.

We base our construction of the solution space on the space  $V_{KP}^2$  of functions of bounded 2 variation  $V^2$  adapted to the three dimensional KP-II equation. This function space will be introduced in more detail in section 2.4. The solution space is defined as

$$\|u\|_{l^q l^p V_{KP}^2} = \left( \sum_{\lambda \in 2^{\mathbb{Z}}} \left( \lambda^{\frac{1}{2}} \sum_{k \in \lambda \cdot \mathbb{Z}^2} \|u_{\Gamma_{\lambda,k}}\|_{V_{KP}^2(\Gamma_{\lambda,k})}^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < +\infty.$$

We need also the homogeneous Fourier restriction space  $\dot{X}^{0,b}$  for  $|b| \leq 1$  which is defined by

$$\|u_1\|_{\dot{X}^{0,b}} = \|\partial_t - \partial_x^3 + \partial_x^{-1} \Delta_y\|^b u_1\|_{L^2} := \|\tau - \omega(\xi, \eta)\|^b \hat{u}_1\|_{L^2} < +\infty$$

for tempered distributions supported in  $[0, \infty) \times \mathbb{R} \times \mathbb{R}^2$ . Here  $\omega(\xi, \eta) = \xi^3 - \frac{|\eta|^2}{\xi}$  is the dispersion function associated to KP-II equation. We define

$$\|u\|_{l^q \dot{X}^{0,b}} = \|\lambda^2 u_{\lambda}(\lambda x, \lambda^2 y, \lambda^3 t)\|_{l_{\lambda}^q \dot{X}^{0,b}} = \left( \sum_{\lambda \in 2^{\mathbb{Z}}} \lambda^{(2-3b)q} \|u_{\lambda}\|_{\dot{X}^{0,b}}^q \right)^{\frac{1}{q}}.$$

Here  $l_{\lambda}^p$  denotes the  $l^p$  norm with respect to the summation over  $\lambda \in 2^{\mathbb{Z}}$ . Finally we define the function space for the fixed point map by

$$\|u\|_X = \|u\|_{l^q l^p V_{KP}^2} + \|u\|_{l^q \dot{X}^{0,b}} < \infty.$$

Since  $\sup_t \|u(t)\|_{L^2} \leq \|u\|_{V_{KP}^2}$  (see [11]) one has  $\sup_t \|u(t)\|_{l^q l^p L^2} \leq \|u\|_X$ . It will be clear from the construction that we obtain solutions in  $u \in C([0, \infty); l^q l^p L^2)$ , for  $1 \leq q < \infty, 1 < p < 2$ . We are ready to state our main results.

**Theorem 1.1.** *For  $1 \leq q < \infty, 1 < p < 2$ , there exists an  $0 < \varepsilon$  such that if  $u_0 \in l^q l^p L^2$  satisfies*

$$\|u_0\|_{l^q l^p L^2} \leq \varepsilon$$

*then there exist a unique global solution  $w$  to (1.1)*

$$w = S(t)u_0 + u$$

*with  $u \in X \subset C(\mathbb{R}, l^q l^p L^2)$ . It satisfies*

$$(1.4) \quad \|u\|_X \leq c \|u_0\|_{l^q l^p L^2}^2.$$

*Here  $S(t)u_0$  is the solution to the homogeneous problem defined by the Fourier transform (see (2.1) in Section 2.1). The flow map*

$$\Phi : B_{\varepsilon} \mapsto X : u_0 \mapsto u \in X$$

*is analytic. Here the symbol  $B_{\varepsilon}$  denotes the ball of radius  $\varepsilon$  in  $l^q l^p L^2$ .*

Scattering is an immediate consequence.

**Corollary 1.2.** *[Scattering] Under the assumption of Theorem 1.1 for  $u_0 \in B_{\varepsilon}$  there exists  $u_{\pm} \in l^q l^p L^2$  such that*

$$u(t) - S(t)u_{\pm} \rightarrow 0 \text{ in } l^q l^p L^2 \text{ as } t \rightarrow \pm\infty.$$

*The wave operators are the inverses of the maps*

$$V_{\pm} : B_{\varepsilon} \ni u_0 \rightarrow u_{\pm} \in l^q l^p L^2.$$

*They are analytic diffeomorphisms to their range if  $\varepsilon$  is sufficiently small.*

*Proof.* It is an important property of the spaces  $V_{KP}^2$  that for  $v \in V_{KP}^2$  the limit

$$\lim_{t \rightarrow \infty} S(-t)v(t)$$

exists. If  $v \in X$  then  $\lim_{t \rightarrow \infty} S(-t)v_{\Gamma_{\lambda,k}}$  exist. But then also

$$\lim_{t \rightarrow \infty} S(-t)v(t)$$

exists in  $l^q l^p L^2$ . Since  $u_0 \rightarrow u(t) \in X$  is analytic also the map  $u_0 \rightarrow \lim_{t \rightarrow \infty} S(-t)u(t)$  is analytic as a function of  $u_0$ . Its derivative at  $u_0 = 0$  is the identity, and hence the map is invertible in a neighborhood of  $u_0 = 0$ .  $\square$

Theorem 1.1 is almost sharp. For  $2 < p < \infty$ , problem (1.1) is ill-posed in the sense that the map  $l^q l^p L^2 \ni u_0 \rightarrow u(t) \in l^q l^p L^2$  cannot be twice differentiable at 0.

**Theorem 1.3.** *Let  $1 \leq q \leq \infty$ ,  $2 < p < \infty$ . Suppose there exists  $T > 0$  and  $\varepsilon > 0$  such that (1.1) admits a unique solution defined on the interval  $[-T, T]$  for initial data in ball of radius  $\varepsilon$  and center 0 in  $l^q l^p L^2$ . Then the flow map*

$$F_t : u_0 \rightarrow u(t)$$

*for (1.1) is not twice differentiable at  $u_0 = 0$  as a map from  $l^q l^p L^2$  to itself.*

We complement the results by studying the relation of the new function spaces to test functions and distributions.

**Theorem 1.4.** *For any  $1 \leq q \leq \infty$ , we have*

- i) *If  $p < 2$  then  $l^q l^p L^2$  embeds continuously into the space of distributions.*
- ii) *If  $p \geq 2$  and  $q > 1$  there is a sequence of Schwartz functions  $\phi_j$  converging to 0 in  $l^q l^p L^2$ , which does not converge in the sense of distributions.*
- iii) *If  $p \leq \frac{4}{3}$ , and  $\phi$  is Schwartz function in  $l^q l^p L^2$  then for all  $y \in \mathbb{R}^2$  we have*

$$\int \phi(x, y) dx = 0.$$

- iv) *The Schwartz functions are contained in  $l^q l^p L^2$  if  $\frac{4}{3} < p < \infty$ .*

**Remark 1.** *For  $l^1 l^2 L^2 = L^2(\mathbb{R}^2; B_{2,1}^{\frac{1}{2}})$  and  $l^2 l^2 L^2 = \dot{H}^{\frac{1}{2},0}$  (see the definition (1.5) below) we do not know whether the flow map is smooth or not.*

It is worthwhile to compare our results to the 2-D KP II initial data problem, which is much better understood. It has the same symmetries - up to obvious changes - as the three dimensional problem. A scaling critical and Galilean invariant space is  $\dot{H}^{-\frac{1}{2},0}$  defined by the norm

$$(1.5) \quad \|u_0\|_{\dot{H}^{s,\sigma}} = \| |\xi|^{-1/2} < \eta >^\sigma \hat{u}_0 \|_{L^2}.$$

In [1], Bourgain settled the global well-posedness of the two dimensional version of (1.1) in  $L^2(\mathbb{R}^2)$ . The assertion was then extended by Takaoka and Tzvetkov [14] (see also Isaza and Mejía [7]) from  $L^2(\mathbb{R}^2)$  to  $H^{s_1, s_2}$  with  $s_1 > -\frac{1}{3}$ ,  $s_2 \geq 0$ . In [13], Takaoka obtained local well-posedness for  $s_1 > -\frac{1}{2}$ ,  $s_2 = 0$  under an additional assumption on the low frequencies which was later removed by Hadac in [3]. Hadac, Herr and the first author [4] studied the two dimensional KP-II equation in the critical case  $s_1 = -\frac{1}{2}$ ,  $s_2 = 0$ . They obtained global well-posedness and scattering result in the homogeneous Sobolev space  $\dot{H}^{-1/2,0}(\mathbb{R}^2)$  with small initial data. A local well posedness result in  $H^{-1/2,0}(\mathbb{R}^2)$  was also obtained in [4]. Some recent results on the KP-II equation can be found in [9].

Much less is known for KP II in three dimensional spaces. Tzvetkov [15] obtained local well-posedness in  $H^s(\mathbb{R}^3)$  with the additional condition  $\partial_x^{-1}u \in H^s(\mathbb{R}^3)$  for  $s > \frac{3}{2}$ . Here  $H^s(\mathbb{R}^3)$  denotes the isotropic Sobolev space. Isaza, López and Mejía [6] constructed unique local solutions in Sobolev space  $H^{s,r}(\mathbb{R}^3)$  defined by the norm

$$\|f\|_{H^{s,r}(\mathbb{R}^3)} := \| \langle \xi \rangle^s \langle \zeta \rangle^r \hat{f}(\zeta) \|_{L_\zeta^2}$$

for  $s, r \in \mathbb{R}$ . Hadac [2] in his Ph.D thesis extended the local well-posedness result to almost all the subcritical cases. He obtained local well posed for (1.1) in  $Y_{s,r}(\mathbb{R}^3)$  for  $s > \frac{1}{2}, r > 0$ . To our best knowledge our result is the first result for initial data in a scaling invariant space, and the first scattering result for the three dimensional problem. Also the bilinear estimates (Proposition 2.1) accounting for dispersion in  $y$  seem to be new.

In the 3-D setting using the vertical direction (i.e. dispersion in the  $y$  variable) is much more important than in the two dimensional problem. This can be seen from the Strichartz estimates in Theorem 2.1 in Section 2.4. In particular the bilinear  $L^4$  estimate by itself seems not to suffice to close the iteration argument, and we need several nontrivial modifications. In particular we use bilinear estimates which give us a gain making use of the dispersion in  $y$  direction. We hope and think that these modifications and the constructions are of interest beyond this particular problem at hand. The 3D-KP II equation may be considered as a problem where the quadratic nonlinearity satisfies a null condition which exactly balances the bilinear estimates and the gain from high modulation, where we are not allowed to lose anything on the  $L^2$  level.

The outline of this paper is following. In Section 2 we prove the Strichartz estimates for the linear equations and a new crucial and fundamental bilinear estimate, Theorem 2.1. In Section 3 we give the proofs of our main results. We first sketch an incorrect heuristic proof to show how far one gets using simple bilinear estimates and high modulation, for  $q = 1$  and  $p = 2$ . A number of estimates is tight in this situation and we have not been able to close the argument for those function spaces. In the remainder of this section we sharpen the bilinear estimates and complete the proof of the main theorem. In Section 4 we complete the paper by a proof of Theorem 1.3 and 1.4.

We use the standard notation  $A \lesssim B$  to mean that there exists constant  $C > 1$  such  $A \leq CB$ . Constants  $C$  may differ from line to line and depend on some obvious indices in the context but not on  $A$  and  $B$ .  $A \sim B$  means  $\frac{1}{C}B \leq A \leq CB$ . Similarly we denote  $A \ll B$  for  $A \leq \frac{1}{C}B$  for some  $C > 0$ . The  $s$  dimensional Hausdorff measure is denoted by  $\mathcal{H}^s$  and its restriction to a set  $S$  by  $\mathcal{H}_S^s$ .

## 2. STRICHARTZ ESTIMATES AND BILINEAR REFINEMENTS

**2.1. Strichartz estimate.** The linear equation

$$u_t + u_{xxx} + \partial_x^{-1}u_{yy} = 0$$

defines a unitary group  $S(t)$  on  $L^2$  by

$$(2.1) \quad \mathcal{F}(S(t)u_0) = e^{it(\xi^3 - |\eta|^2/\xi)} \hat{u}_0.$$

Given  $u_0$  the solution  $u(t) = S(t)u_0$  satisfies the Strichartz estimates of the next lemma. We denote by  $|D_x|^s$  the Fourier multiplier  $|\xi|^s$ ,  $\xi$  being as always the Fourier variable of  $x$ .

**Lemma 2.1.** *Suppose that  $2 \leq p \leq \infty$  and*

$$(2.2) \quad \frac{2}{p} + \frac{3}{q} = \frac{3}{2}.$$

*Then the following estimate holds for all  $u_0 \in \mathcal{S}$*

$$\|u\|_{L_t^p L_x^q} \lesssim \| |D_x|^{\frac{1}{3p}} u_0 \|_{L^2}.$$

*If  $2 \leq q < \infty$*

$$(2.3) \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}$$

*then*

$$\|u\|_{L_t^p L_x^q} \lesssim \| |D_x|^{\frac{2}{p}} u_0 \|_{L^2}.$$

*Proof.* We only sketch the proof. By a Littlewood Paley decomposition (see (1.2)) and Hölder's inequality the estimate follows from

$$\|u_1\|_{L_t^p L_x^q} \leq c \|u_1(0)\|_{L^2}$$

for Strichartz pairs  $(p, q)$  which in turn is a consequence of the calculation of the complex Gaussian (as oscillatory integral)

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} e^{iy \cdot \eta - it\eta^2 / \xi + it\xi^3} d\eta = \frac{\xi}{4ti} e^{i\frac{\xi|y|^2}{4t} + it\xi^3}.$$

By stationary phase and the lemma of van der Corput we obtain

$$\left| \int \frac{\xi}{|\xi|} |\xi|^{1/2} e^{i(x + \frac{|y|^2}{4t})\xi + it\xi^3} d\xi \right| \leq C|t|^{-\frac{1}{2}}$$

which we write as

$$\|D_x^{\frac{1}{2}} \mathcal{F}^{-1} e^{it(\xi^3 - \eta^2/\xi)}\|_{sup} \leq C|t|^{-\frac{3}{2}}.$$

By complex interpolation, the Hardy-Littlewood-Sobolev resp. weak Young inequality and a  $T^*T$  argument (2.2) follows. The endpoint  $p = 2$  and  $q = 6$  follows from [8].

The estimate

$$\left| \int_{1 \leq |\xi| \leq 2} \xi e^{i(x + \frac{|y|^2}{4t})\xi + it\xi^3} d\xi \right| \leq C$$

is trivial. It leads to the second estimate (2.3) by the same standard arguments.  $\square$

It is remarkable that there is so much flexibility in the choice of  $p$  and  $q$ . This is true for the Schrödinger group, but there it comes from a trivial combination of (sharp) Strichartz estimates with Sobolev embedding. Here the situation is different due to the unbounded  $y$  direction.

**2.2. Bilinear estimates.** There is an important special case of (2.3):

$$(2.4) \quad \|u\|_{L^4(\mathbb{R}^4)} \leq c \| |D_x|^{\frac{1}{2}} u_0 \|_{L^2(\mathbb{R}^3)}.$$

The proof of the main theorem relies crucially on the following bilinear refinements. We denote by  $u_{<\mu}$  the Fourier projection to all  $\xi$  frequencies less in absolute value than  $\mu$ , by  $u_{>\lambda}$  the Fourier projection to  $\xi$  frequencies with absolute value  $> \lambda$  and by  $u_{\mu,\Gamma}$  the Fourier projection to

$$\left\{ (\xi, \eta) : \mu < |\xi| \leq 2\mu, \frac{\eta}{\xi} \in \mu\Gamma \right\}.$$

Let  $|\Gamma|$  denote the Lebesgue measure of  $\Gamma$ . With this notation the following variant or sharpening of the bilinear estimate is true.

**Theorem 2.1.** *Let  $0 < \mu, \lambda$ . Then*

$$(2.5) \quad \|u_{<\mu} v_{>\lambda}\|_{L^2} \leq c\mu \|u_0\|_{L^2} \|v_0\|_{L^2},$$

and, if  $\mu \leq \lambda$ , if  $\Gamma \subset \mathbb{R}^2$  is measurable, and if either

- $\mu \leq \lambda/8$  or
- $\lambda/8 < \mu \leq \lambda$  and  $\Gamma \subset B_\lambda(0)$  and the support of the Fourier transform of  $v_\lambda$  is disjoint from  $\mathbb{R} \times \mathbb{R} \times B_{10\lambda^2}(0)$

then

$$(2.6) \quad \left\| \int_{\mathbb{R} \times \mathbb{R}^2} \left( \lambda + \left| \frac{\eta_1}{\xi_1} - \frac{\eta - \eta_1}{\xi - \xi_1} \right| \right) \hat{u}_{\mu, \Gamma}(t, \xi_1, \eta_1) \hat{v}_\lambda(t, \xi - \xi_1, \eta - \eta_1) d\xi_1 d\eta_1 \right\|_{L^2} \\ \lesssim \mu |\Gamma|^{\frac{1}{2}} \|u_{0, \mu, \Gamma}\|_{L^2} \|v_{0, \lambda}\|_{L^2}.$$

**Remark 2.** Here as always  $u_{0, \mu, \Gamma}$  denotes the Fourier projection of the initial data.

**Remark 3.** The condition for the second inequality is needed for a bound of a derivative from below at a single point in the argument in (2.13) below. If  $\mu \sim \lambda$ ,  $\Gamma = B_\lambda(0)$  and the Fourier support of  $v_\lambda$  is contained in  $\mathbb{R} \times \mathbb{R} \times B_{10\lambda^2}(0)$  then there is no gain compared to the Strichartz estimate (2.4).

*Proof.* We consider solutions to the dispersive equation

$$(2.7) \quad i\partial_t u + \phi(D)u = 0$$

with  $\phi(D)$  defined as Fourier multiplier with a smooth real function  $\phi$ . Then the Fourier transform of a solution with initial data  $u_0$  is a complex measure supported on the characteristic set  $\{(\tau, \xi) : \tau = \phi(\xi)\}$ . Here we denote all spatial Fourier variables by  $\xi$ . If  $u$  is the solution to (2.7) with initial data  $u_0$  then (essentially using a regularization and the coarea formula to make sense of the calculus of Dirac measures)

$$\hat{u} = \hat{u}_0(\xi) \delta_\Phi = \sqrt{2\pi} (1 + 2|\nabla \phi|^2)^{-1/2} \hat{u}_0(\xi) d\mathcal{H}^d|_\Sigma$$

where  $\Sigma = \{(\tau, \xi) : \tau = \phi(\xi)\}$  is the characteristic set, and

$$\|\hat{u}\|_{L^2(\delta_\Phi)} = (2\pi)^{-1/2} \|u_0\|_{L^2}.$$

By the formula of Plancherel bilinear estimates for dispersive equations are equivalent to  $L^2$  estimates of convolutions of such signed measures supported in such surfaces. By the Cauchy-Schwarz inequality and the theorem of Fubini, for non-negative bounded measurable functions  $h$  and  $l$ ,

$$\begin{aligned} & \|fh * gl\|_{L^2(\mathbb{R}^d)}^2 \\ &= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} f(x) (h(x)l(z-x))^{1/2} g(z-x) (h(x)l(z-x))^{1/2} dx \right)^2 dz \\ &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f^2(x) h(x) l(z-x) dx \int_{\mathbb{R}^d} g^2(y) h(z-y) l(y) dy dz \\ &\leq \int_{\mathbb{R}^{2d}} \left[ \int_{\mathbb{R}^d} h(z-y) l(z-x) dz \right] f^2(x) h(x) g^2(y) l(y) dx dy. \end{aligned}$$

Suppose that  $U, V \subset \mathbb{R}^d$  are open,  $\Phi_1 \in C^1(U)$ ,  $\Phi_2 \in C^1(V)$  and that the gradients  $\nabla \Phi_i$  are nonzero where  $\Phi_i$  vanishes. We define the Dirac measures  $\delta_{\Phi_i}$  by approximation. The zero set of  $\Phi_i$  is denoted by  $\Sigma_i$ . The calculation above yields

$$\|f\delta_{\Phi_1} * g\delta_{\Phi_2}\|_{L^2} \leq C\|f\|_{L^2(\delta_{\Phi_1})}\|g\|_{L^2(\delta_{\Phi_2})}$$

where

$$(2.8) \quad C^2 = \sup_{x \in \Sigma_1, y \in \Sigma_2} \int_{\mathbb{R}^d} \delta_{\Phi_2}(z-x) \delta_{\Phi_1}(z-y) dz$$

which has again to be understood as limit through the approximation of the Dirac measures by smooth functions. By the coarea formula the integral can be rewritten. Let

$$\Sigma_{x,y} = \{z \in \mathbb{R}^d : z-x \in \Sigma_2, z-y \in \Sigma_1\} = (x + \Sigma_2) \cap (y + \Sigma_1).$$

With

$$D = \begin{pmatrix} d\Phi_1(z-x) \\ d\Phi_2(z-y) \end{pmatrix}$$

$$J(z, x, y) = (\det(D^T D))^{1/2},$$

we have

$$(2.9) \quad C^2 = \sup_{x,y} \int_{\Sigma_{x,y}} J(x, y, z) d\mathcal{H}^{d-2}(z).$$

The case  $\Phi_i(\tau, \xi) = \tau - \phi(\xi)$ , but with  $\Phi_1$  defined on  $\mathbb{R} \times A$  and  $\Phi_2$  on  $\mathbb{R} \times B$  is of particular interest. Integrating out  $\tau$  (2.8) simplifies to (with  $n = d - 1$ )

$$(2.10) \quad C^2 = \sup_{\xi_1 \in A, \xi_2 \in B} \sup_{\tau} \int_{\mathbb{R}^n} \delta_{\phi(\xi-\xi_1)-\phi(\xi-\xi_2)-\tau} d\xi.$$

The first case of interest is  $U = \{(\tau, \xi, \eta) : |\xi| \leq \mu\}$ ,  $V = \{(\tau, \xi, \eta) : \lambda \leq |\xi|\}$  and

$$\phi = \phi_1 = \phi_2 = \xi^3 - |\eta|^2/\xi.$$

To obtain the bilinear estimate (2.5) we have to estimate the integrals in (2.10) by a constant times  $\mu^2$ . By the  $L^4$  estimate (2.4) we may assume that  $\mu \leq \lambda/2$  and estimate the quantity in (2.8):

$$(2.11) \quad C^2 = \sup_{\tau, \xi_1, \xi_2, \eta_1, \eta_2} \int_{\mathbb{R}^3, \xi-\xi_2 \in A, \xi-\xi_1 \in B} \delta_{\phi(\xi-\xi_1, \eta-\eta_1)-\phi(\xi-\xi_2, \eta-\eta_2)-\tau} d\eta d\xi.$$

The algebraic identity

$$(2.12) \quad \begin{aligned} & \phi(\xi - \xi_1, \eta - \eta_1) - \Phi(\xi - \xi_2, \eta - \eta_2) + \Phi(\xi_1 - \xi_2, \eta_1 - \eta_2) \\ & \quad + 3(\xi_1 - \xi_2)(\xi - \xi_1)(\xi - \xi_2) \\ & = (\xi_2 - \xi_1)(\xi - \xi_1)(\xi - \xi_2) \left( \frac{\left| \frac{\eta - \eta_1}{\xi - \xi_1} - \frac{\eta - \eta_2}{\xi - \xi_2} \right|}{|\xi_1 - \xi_2|} \right)^2 \end{aligned}$$

can be verified by an easy calculation. In particular, if we fix  $\xi$  then either the  $\eta$  integral is over the empty set, a point, or it is an integral over a circle, in which case by (2.12) (it suffices to consider the coefficient of the quadratic term since the integral is independent of the radius)

$$\int_{\mathbb{R}^2} \delta_{\phi(\xi-\xi_1, \eta-\eta_1)-\phi(\xi-\xi_2, \eta-\eta_2)-\tau} d\eta = \frac{4\pi|\xi_2 - \xi_1|}{|\xi - \xi_1||\xi - \xi_2|}$$



and we estimate the integral with respect to  $\xi$  for  $\mu \leq \lambda/2$

$$\frac{2\pi}{|\xi_2 - \xi_1|} \int_{|\xi - \xi_2| \leq \mu} |\xi - \xi_2| |\xi - \xi_1| d\xi \leq 8\pi\mu^2.$$

Together with the  $L^4$  Strichartz estimate this implies estimate (2.5).

We turn to the second part, (2.6), for which we repeat the calculus argument. Here we want to recover the stronger bilinear estimate for the KP equation where one gains a full derivative. Of course this can only be done by reducing the domain of the integration. The final integration then leads to the factor given by measure of  $|\Gamma|$ .

Let  $\Phi_i$  be as above. Instead of estimating the convolution itself we claim that

$$\begin{aligned} & \left\| \int h(y, x-y) f_1(y) f_2(x-y) \delta_{\Phi_1}(y) \delta_{\Phi_2}(x-y) dy \right\|_{L^2} \\ & \leq C \|f_1\|_{L^2(\delta_{\Phi_1})} \|f_2\|_{L^2(\delta_{\Phi_2})} \end{aligned}$$

where

$$C^2 = \sup_{x \in \Sigma_1, y \in \Sigma_2} \int h^2(z-x, z-y) \delta_{\Phi_1}(z-x) \delta_{\Phi_2}(z-y) dz.$$

This follows by the same calculation as above.

We take up the bilinear estimate for the KPII equation and estimate the integral in (2.11) with the integration restricted to a suitable set. We fix  $\tau, \xi_1, \xi_2, \eta_1$  and  $\eta_2$ . We search an estimate which contains the measure of  $\Gamma$  and apply the transformation formula and Fubini's theorem to take the integration with respect to  $\Gamma$  as outer integration. This yields the desired estimate provided we get uniform bounds for the integral with respect to  $\xi$  for  $\frac{\eta - \eta_2}{\xi - \xi_2} = \rho \in \mathbb{R}^2$  fixed. The Jacobian determinant of the map

$$(\xi, \eta) \rightarrow \left( \xi, \frac{\eta - \eta_2}{\xi - \xi_2} \right)$$

from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  is  $\frac{1}{|\xi - \xi_2|^2}$ . We assume that one of the conditions of the second part of the theorem holds. Let  $h = \lambda + \left| \frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2} \right|$  be the integrand to be studied. We recall that  $\Gamma \subset \mathbb{R}^2$  and denote

$$B = \left\{ (\xi, \eta) : \mu/2 \leq |\xi| \leq \mu, \frac{\eta - \eta_2}{\xi - \xi_2} \in \mu\Gamma \right\}.$$

Then

$$\begin{aligned} & \int_B \left( \lambda + \left| \frac{\eta - \eta_1}{\xi - \xi_1} - \frac{\eta - \eta_2}{\xi - \xi_2} \right| \right)^2 \delta_{\phi(\xi - \xi_1, \eta - \eta_1) - \phi(\xi - \xi_2, \eta - \eta_2)} d\xi d\eta \\ & = \int_{\Gamma} \int \left( \lambda + \left| \frac{\eta - \eta_1}{\xi - \xi_1} - \frac{\eta - \eta_2}{\xi - \xi_2} \right| \right)^2 |\xi - \xi_2|^2 \delta_{g_\rho}(\xi) d\xi d\gamma \\ & \leq C\mu^2 |\Gamma| \end{aligned}$$

where we calculated with

$$\begin{aligned} & (\xi - \xi_1)^3 - \frac{(\eta - \eta_1)^2}{\xi - \xi_1} - (\xi - \xi_2)^3 + \frac{(\eta - \eta_2)^2}{\xi - \xi_1} \\ & = (\xi - \xi_1)^3 - \frac{(\rho \cdot (\xi - \xi_2) + \eta_2 - \eta_1)^2}{\xi - \xi_1} - (\xi - \xi_2)^3 + (\xi - \xi_2)|\rho|^2 \\ & =: g_\rho(\xi). \end{aligned}$$

Clearly  $g_\rho(\xi) = \tau$  if and only if

$$(\xi - \xi_1)^4 - (\rho \cdot (\xi - \xi_2) + \eta_2 - \eta_1)^2 - (\xi - \xi_2)^3(\xi - \xi_1) + (\xi - \xi_1)(\xi - \xi_2)|\rho|^2 = \tau$$

and hence there are at most 4 values of  $\xi$  where  $g_\rho = \tau$ . Moreover

$$\begin{aligned} \left| \frac{d}{d\xi} g(\xi) \right| &= \left| 3(\xi - \xi_1)^2 - 2\rho \frac{\rho(\xi - \xi_2) + \eta_2 - \eta_1}{\xi - \xi_1} \right. \\ &\quad \left. + \frac{(\rho(\xi - \xi_2) + \eta_2 - \eta_1)^2}{(\xi - \xi_1)^2} - 3(\xi - \xi_2)^2 + |\rho|^2 \right| \\ (2.13) \quad &= \left| 3(\xi - \xi_1)^2 - 3(\xi - \xi_2)^2 + \left| \frac{\eta - \eta_2}{\xi - \xi_2} - \frac{\eta - \eta_1}{\xi - \xi_1} \right|^2 \right| \\ &\sim \left( \lambda + \left| \frac{\eta - \eta_1}{\xi - \xi_1} - \frac{\eta - \eta_2}{\xi - \xi_2} \right| \right)^2 \end{aligned}$$

since  $g_\rho(\xi) = \tau$  at most at four points, and it satisfies the lower bound there.  $\square$

**2.3. Functions of bounded  $p$  variation and their predual.** Functions of bounded  $p$  variation were introduced by N.Wiener [16]. The space of function of bounded  $p$  variation and their pre-dual spaces  $U^p$  were defined by D.Tataru and the first author of this paper in [10].  $V_{KP}^p$  and  $U_{KP}^p$  are defined by  $S(t)V^p$  and  $S(t)U^p$ . Here  $S(t)$  is the unitary group defined in (2.1). We refer the reader to [4] for the following statements and further properties about  $U_{KP}^p$  and  $V_{KP}^p$ . Let  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $1 < p < \infty$ . The duality pairing can formally be written as

$$B(u, v) = \int v(\partial_t + \partial_{xxx} - \partial_x^{-1} \Delta_y) \bar{u} dx dy dt,$$

but a correct definition requires more care (see [5]). The space  $V_{KP}^{p'}$  is the dual of  $U_{KP}^p$  with respect to this duality pairing. We denote by  $V_{rc}^p$  the subspace of  $V_{KP}^p$  of right continuous functions with limit 0 at  $-\infty$ .

The spaces  $U^p$  have an atomic structure and the Strichartz estimates imply

$$(2.14) \quad \|u\|_{L^p L^q} \leq c_1 \| |D_x|^{\frac{1}{3p}} u \|_{U_{KP}^p}$$

where  $\frac{2}{p} + \frac{3}{q} = \frac{3}{2}$ ,  $2 \leq p, q \leq \infty$  and

$$(2.15) \quad \|u\|_{L^p L^q} \leq \|D^{\frac{1}{p}} u\|_{U_{KP}^p}$$

where  $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$ ,  $2 < p \leq \infty$ . Moreover one has the inclusions

$$(2.16) \quad \|u\|_{U_{KP}^p} \leq c \|u\|_{V_{KP}^q}$$

whenever  $q < p$  and  $u \in V_{KP}^q$  is right continuous. Similarly we obtain from the bilinear estimates of Theorem 2.1 under the same assumptions there,

$$(2.17) \quad \|u_\mu v_\lambda\|_{L^2} \leq c\mu \|u_\mu\|_{U_{KP}^2} \|v_\lambda\|_{U_{KP}^2}$$

and

$$(2.18) \quad \left\| \int_S \left( \lambda + \left| \frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2} \right| \right) \hat{u}_{\mu, \Gamma} \hat{v}_\lambda \right\|_{L^2} \lesssim \mu |\Gamma|^{\frac{1}{2}} \|u_{\mu, \Gamma}\|_{U_{KP}^2} \|v_\lambda\|_{U_{KP}^2}.$$

The  $V_{KP}^2$  spaces behave well with respect to further decompositions:

$$(2.19) \quad \|u_\lambda\|_{V_{KP}^2} \leq \|u_\lambda\|_{l^2 V_{KP}^2},$$

see [11]. They allow the following decomposition

**Lemma 2.2.** *Suppose that  $1 < p < q < \infty$ . There exists  $\delta > 0$  so that for any right continuous  $v \in V_{KP}^p$  and  $M > 1$  there exists  $u \in U_{KP}^p$  and  $w \in U_{KP}^q$  such that*

$$v = u + w$$

$$\|u\|_{U_{KP}^p} \leq M, \quad \|w\|_{U_{KP}^q} \leq e^{-\delta M}.$$

From (2.17), the  $L^4$  Strichartz estimates and logarithmic interpolation lemma 2.2 (see again [4]), we obtain for any  $0 < \varepsilon \ll 1$ ,

$$(2.20) \quad \|u_\mu v_\lambda\|_{L^2} \leq C(\varepsilon) \mu \left(\frac{\lambda}{\mu}\right)^\varepsilon \|u_\mu\|_{V_{KP}^2} \|v_\lambda\|_{V_{KP}^2}.$$

Similarly the bilinear estimate (2.6) implies bilinear estimates with respect to  $U_{KP}^2$ , and via logarithmic interpolation, estimate with respect to the  $V_{KP}^2$  norm.

Later we will make use of the spaces  $U_{KP}^1 \subset V_{KP}^1$  which carry identical norms, which, for functions given by  $S(-t)u(t) = \int_{-\infty}^t f(s)ds$  is  $\int_{\mathbb{R}} |f|dt$ . We define

$$\begin{aligned} \|v\|_{V_{KP}^1} &= \|S(-t)v(t)\|_{BV} \\ &= \sup_{t_0 < t_2 < \dots < t_n} \sum_{j=1}^n \|S(-t_j)v(t_j) - S(-t_{j-1})v(t_{j-1})\|_{L^2} \end{aligned}$$

where we allow  $t_n = \infty$  (recall the convention  $v(\infty) = 0$ ). We denote by  $U_{KP}^1$  the Banach space of all right continuous functions with  $\lim_{t \rightarrow -\infty} u(t) = 0$  for which this norm is finite. It is not hard to see that

$$\|u\|_{U_{KP}^1} = \|S(-t)u(t)\|_{BV(\mathbb{R}, L^2)}$$

Then  $U_{KP}^1 \subset U_{KP}^2$ . We will use an improvement of the estimate for high modulation. Let  $\Phi \in \mathcal{S}(\mathbb{R})$  with  $\hat{\Phi} = 1$  for  $|\tau| \leq 1$ ,  $\hat{\Phi} = 0$  for  $|\tau| \geq 2$ . Then, for  $f$  with  $f' \in L^1$

$$\begin{aligned} \|f - \Phi * f\|_{L^1} &= \left\| \int (f(t) - f(s))\Phi(t-s)ds \right\|_{L^1} \\ &\leq \int_{\mathbb{R}} |\Phi(\sigma)| \int |f(t) - f(t-\sigma)| dt d\sigma \\ &= \int_{\mathbb{R}} |\sigma| |\Phi(\sigma)| d\sigma \int |f'(t)| dt. \end{aligned}$$

Rescaling and an approximation yield the high modulation estimate

$$(2.21) \quad \|u_\lambda^{>\Lambda}\|_{L_t^1 L^2} \leq c\Lambda^{-1} \|u_\lambda\|_{U_{KP}^1}.$$

Here  $u^{>\Lambda}$  resp  $u^{\leq\Lambda}$  means the Fourier projection to high resp- low modulation, i.e. to

$$|\tau - \omega(\xi, \eta)| := \left| \tau - \left( \xi^3 - \frac{|\eta|^2}{\xi} \right) \right| > \Lambda$$

resp.  $\leq \Lambda$ . By the definition of the Fourier restriction spaces

$$\|u^{>\Lambda}\|_{L^2} \leq \Lambda^{-b} \|u^{>\Lambda}\|_{\dot{X}^{0,b}}, \quad \|u^{>\Lambda}\|_{L^2} \leq \Lambda^{-1/2} \|u^{>\Lambda}\|_{V_{KP}^2},$$

and similarly

$$\|u^{\sim\Lambda}\|_{U_{KP}^2} \leq \Lambda^{1/2} \|u\|_{L^2}.$$

see [4].

**2.4. A bilinear operator.** The bilinear estimates of Theorem 2.1 state some off-diagonal decay in the bilinear terms. This suggests to decompose waves into wave packets of corresponding Fourier support. We recall that we partition  $\{\lambda, 2\lambda\} \times \mathbb{R}^2$  into sets  $\Gamma_{\lambda,k}$  (1.3). Theorem 2.1 effectively diagonalizes the bilinear estimate in the sector determined by the large frequency. To capture this we define

$$\Gamma_{\lambda,k,L} = \left\{ (\xi_1, \eta_1) : \lambda \leq \xi_1 \leq 2\lambda, \left| \frac{\eta_1}{\xi_1} - kL \right|_\infty \leq \frac{L\lambda}{2} \right\}$$

and  $\Gamma_{\mu,k,L\lambda/\mu}$  is the set in frequency  $|\xi| \sim \mu$  which corresponds to  $\Gamma_{\lambda,k}$  in the bilinear estimate of Theorem 2.1. We define a smooth bilinear projection which is compatible with scaling and the Galilean symmetry. Here we again denote the Fourier transform in space time by  $\mathcal{F}$  resp.  $\hat{\cdot}$ . Let  $\phi_1 \in C_0^\infty((-129, 129) \times (-129, 129))$ , identically 1 in  $(-128, 128) \times (-128, 128)$  and even. We define for  $L = 2^k$  with  $k \geq 1$

$$\psi_L(s) = \phi_1(s/L) - \phi_1(2s/L)$$

and

$$\rho_L(\xi_1, \eta_1, \xi_2, \eta_2) := \psi_L \left( \frac{\frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2}}{\xi_1 + \xi_2} \right).$$

For  $L = 1$ , we make the modification

$$\rho_1(\xi_1, \eta_1, \xi_2, \eta_2) := \phi_1 \left( \frac{\frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2}}{\xi_1 + \xi_2} \right).$$

**Definition 2.3.** We define the bilinear operators by their Fourier transform

$$\mathcal{F}(T_L(v_\mu, u_\lambda))(\tau, \xi, \eta) = \int_S \rho_L(\xi_1, \eta_1, \xi_2, \eta_2) \hat{v}_\mu(\tau_1, \xi_1, \eta_1) \hat{u}_\lambda(\tau_2, \xi_2, \eta_2) d\mathcal{H}^4.$$

Here  $S = \{\xi = \xi_1 + \xi_2, \eta = \eta_1 + \eta_2, \tau = \tau_1 + \tau_2\}$  and  $d\mathcal{H}^4$  denotes the 4-Dimensional Hausdorff measure on it.

The product is the dyadic sum of these bilinear operators. The key properties of the bilinear projection are its symmetry, and the bounds of Proposition 2.5 below.

**Lemma 2.4.** The following symmetry identity always holds.

$$(2.22) \quad \begin{aligned} \int u_\lambda T_L(v_\mu, w_\mu) dx dy dt &= \int v_\lambda T_L(u_\lambda, w_\mu) dx dy dt \\ &= \int w_\mu T_L(u_\lambda, v_\lambda) dx dy dt. \end{aligned}$$

*Proof.* This follows from the algebraic calculation

$$\frac{\frac{\eta_1 + \eta_2}{\xi_1 + \xi_2} - \frac{\eta_1}{\xi_1}}{\xi_2} = \frac{\frac{\eta_2}{\xi_2} - \frac{\eta_1}{\xi_1}}{\xi_1 + \xi_2}.$$

□

The following bilinear estimates provide us with a crucial new tool. Below the index  $\{\cdot\}_+$  denotes the positive part.

**Proposition 2.5.** Let  $\varepsilon > 0$ ,  $1 \leq p, q, r \leq \infty$  with

$$\frac{1}{r} \leq \frac{1}{p} + \frac{1}{q}$$

and  $L \in 2^k, k = 0, 1, 2, \dots$ . Then the following estimates hold

$$(2.23) \quad \|T_L(u_\mu, v_\lambda)_\lambda\|_{l^r L^2} \leq C\mu \left(\frac{L\lambda}{\mu}\right)^{1-\frac{2}{p}+\varepsilon} L^{(1-\frac{2}{q})_+ + (\frac{2}{r}-1)_+} \|u_\mu\|_{l^p V_{KP}^2} \|v_\lambda\|_{l^q V_{KP}^2}$$

and

$$(2.24) \quad \begin{aligned} & \| (T_L(u_\lambda, v_\lambda))_\mu \|_{l^r L^2} \\ & \leq C\lambda \left(\frac{L\lambda}{\mu}\right)^{(\frac{2}{r}-1)_+} L^{(1-\frac{2}{p}) + (1-\frac{2}{q})_+ + \varepsilon} \|u_\lambda\|_{l^p V_{KP}^2} \|v_\lambda\|_{l^q V_{KP}^2}. \end{aligned}$$

*Proof.* We consider the case  $\mu < \lambda/4$  for (2.23) first. By rescaling we may assume that  $\mu = 1 < \lambda/4$ . We decompose the bilinear term further, using that by the definition of  $T_L$  there is only a contribution if

$$\left| \frac{\frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2}}{\xi_1 + \xi_2} \right| \sim L.$$

It is important that this relation is equivalent to

$$\left| \frac{\frac{\eta_1 + \eta_2}{\xi_1 + \xi_2} - \frac{\eta_2}{\xi_2}}{\xi_1} \right| \sim L.$$

Since  $1 \leq \lambda/4$  we have  $|\xi_2| \sim |\xi_1 + \xi_2| \sim \lambda$  and both  $\xi_2$  and  $\xi_1 + \xi_2$  have the same sign. For simplicity we assume that both are positive. Recall that  $|\xi_1| \sim 1$ . We begin with the case  $L = 1$  resp.

$$\left| \frac{\frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2}}{\xi_1 + \xi_2} \right| \leq 1.$$

If  $(\xi_2, \eta_2) \in \Gamma_{\lambda, k}$  then the  $l^r$  summation in (2.23) over  $\Gamma_{\lambda, l}$  contributes only if  $|k - l| \leq C$ . We simplify our lives and restrict to  $l = k$ . The situation is similar if  $L > 1$ . and we obtain the restriction that the indices are of distance  $\sim 1$  and the slopes have distance  $\sim L\lambda$ .

Hence, by the same abuse of notation as usual, and with the sets  $\Gamma_{1, k, \lambda}$  defined at the beginning of this subsection

$$(2.25) \quad (T_1(u_1, v_\lambda))_{\Gamma_{\lambda, k}} = (T_1(u_{\Gamma_{1, k, \lambda}}, v_{\Gamma_{\lambda, k}}))_{\Gamma_{\lambda, k}}.$$

We search for an  $L^2$  estimate and ignore the outer restriction to  $\Gamma_{\lambda, k}$  in the notation. By the bilinear estimate we get

$$\|u_{\Gamma_{1, l}} v_{\Gamma_{\lambda, k}}\|_{L^2} \leq c \frac{1}{\lambda} \|u_{\Gamma_{1, l}}\|_{U_{KP}^2} \|v_{\Gamma_{\lambda, k}}\|_{U_{KP}^2}.$$

There are  $\sim \lambda^2$  such terms in  $u_{\Gamma_{1, k, \lambda}}$  contributing to the sum and hence by Hölder's inequality applied to the finite sum

$$\|u_{\Gamma_{1, k, \lambda}} v_{\Gamma_{\lambda, k}}\|_{L^2} \leq c \frac{1}{\lambda} \lambda^{2-\frac{2}{p}} \|u_{\Gamma_{1, k, \lambda}}\|_{l^p U_{KP}^2} \|v_{\Gamma_{\lambda, k}}\|_{U_{KP}^2}.$$

The  $L^4$  Strichartz estimate gives

$$\begin{aligned} \|u_{\Gamma_{1,k,\lambda}} v_{\Gamma_{\lambda,k}}\|_{L^2} &\leq \sum_{\Gamma_{1,l} \subset \Gamma_{1,k,\lambda}} \|u_{\Gamma_{1,l}} v_{\Gamma_{\lambda,k}}\|_{L^2} \\ &\leq c \sum_{\Gamma_{1,l} \subset \Gamma_{1,k,\lambda}} \lambda^{\frac{1}{2}} \|u_{\Gamma_{1,l}}\|_{U_{KP}^4} \|v_{\Gamma_{\lambda,k}}\|_{U_{KP}^4} \\ &\leq c \lambda^{\frac{1}{2}} \lambda^{2-\frac{2}{p}} \|u_{\Gamma_{1,k,\lambda}}\|_{l^p U_{KP}^4} \|v_{\Gamma_{\lambda,k}}\|_{U_{KP}^4}. \end{aligned}$$

where the summation is with respect to those  $l$  for which  $\Gamma_{1,l} \subset \Gamma_{1,k,\lambda}$ . With the logarithmic interpolation of Lemma 2.2 we arrive at

$$\|u_{\Gamma_{1,k,\lambda}} v_{\Gamma_{\lambda,k}}\|_{L^2} \leq c \lambda^{1-\frac{2}{p}+\varepsilon} \|u_{\Gamma_{1,k,\lambda}}\|_{l^p V_{KP}^2} \|v_{\Gamma_{\lambda,k}}\|_{V_{KP}^2}.$$

The summation with respect to  $k$  is trivial and we arrive at the first estimate (2.23), also for  $L > 1$ , for which there are only the obvious modifications, up to an explanation why we may simply drop the operator  $T_L$  once we restricted the support of the Fourier transforms of the factors. Bounded spatial Fourier multipliers define bounded operators on the function spaces  $U_{KP}^p$  and  $V_{KP}^p$ . Our problem is that  $T_L$  is a bilinear Fourier multiplier, and we have to reduce the estimates to estimates of Fourier multipliers acting on single functions. We recall that

$$\rho_L(\xi_1, \eta_1, \xi_2, \eta_2) := \psi_L \left( \frac{\frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2}}{\xi_1 + \xi_2} \right)$$

and we want to bound  $T_L(u_{\Gamma_{\mu,k,L\lambda/\mu}}, u_{\Gamma_{\lambda,k',L}})$  which is zero unless  $4 \leq |k - k'|_\infty \leq 20$ . Without loss of generality we consider  $64 \leq k_1 - k'_1 \leq 1000$ . We apply a Galilee transform which reduces the problem to  $k_1 + k'_1 = 0$ ,  $k_2 = 0$  and  $|k'_2| \leq 20$ . More precisely we expand

$$(2.26) \quad u_{\Gamma_{\mu,k,L\lambda/\mu}} = \sum_{l \in A} u_{\Gamma_{\mu,l}}$$

where  $A$  is set of cardinality  $(L\lambda/\mu)^2$ . The function  $\rho_L$  is a smooth function on  $\Gamma_{\mu,k,L\lambda/\mu} \times \Gamma_{\lambda,k',L}$ . We choose a smooth extension supported in

$$\begin{aligned} &((-3\mu, -\frac{4}{3}\mu) \cup (\frac{4}{3}\mu, 3\mu)) \cup \{|\eta - kL\lambda\mu|_\infty \leq L\lambda\mu\} \\ &\times ((-3\lambda, -\frac{4}{3}\lambda) \cup (\frac{4}{3}\lambda, 3\lambda)) \cup \{|\eta - kL\lambda^2|_\infty \leq L\lambda^2\}, \end{aligned}$$

which, by an abuse of notation, we call again  $\rho_L$ . Its derivative satisfies

$$\left| \partial_{\xi_1}^k \partial_{\eta_1}^\alpha \partial_{\xi_2}^l \partial_{\eta_2}^\beta \psi_1 \left( \frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2} \right) \right| \leq c \mu^{-k} \lambda^{-l} (L\mu\lambda)^{-|\alpha|} (L\lambda^2)^{-|\beta|}.$$

We expand it into a fast converging Fourier series and we multiply it by a suitable smooth product cutoff function

$$\begin{aligned} \rho_L &= \sum_{\alpha} \rho_1(\xi_1/\mu) e^{2\pi i \alpha_1 \xi_1/\mu} \rho_2(\eta_1/L\lambda\mu) e^{2\pi i \eta_1 \alpha_2/(L\lambda\mu)} \\ &\quad \times \rho_3(\xi_2/\lambda) e^{2\pi i \alpha_3 \xi_2/\lambda} \rho_4(\eta_2/(L\lambda^2)) e^{2\pi i \eta_2/(L\lambda^2)} \\ &=: f^\alpha = \sum_{\alpha} a_\alpha f_1^\alpha(\xi_1) f_2^\alpha(\eta_1) f_3^\alpha(\xi_2) f_4^\alpha(\eta_2) \end{aligned}$$

with uniform bounded compactly supported functions  $f_j^\alpha$  and summable coefficients  $a^\alpha$ . It suffices to bound the operator

$$\begin{aligned} T_{f^\alpha}(u_{\mu,k,L\lambda/\mu}, v_{\Gamma_{\lambda,k',L}}) &= M_{f_1^\alpha f_2^\alpha} u_{\mu,k,L\lambda/\mu} M_{f_3^\alpha f_4^\alpha} v_{\Gamma_{\lambda,k',L\lambda^2}} \\ &= \tilde{u}_{\Gamma_{\mu,k,L\lambda/\mu}} \tilde{v}_{\Gamma_{\lambda,k',L\lambda^2}}. \end{aligned}$$

where  $M_f$  denotes the Fourier multiplier. The bilinear estimate above, together with the observation that spatial Fourier multipliers define bounded operators on  $U_{KP}^p$  and  $V_{KP}^p$  completes the argument for the first estimate (2.23) if  $\mu \leq \lambda/4$ . If  $\mu > \lambda/4$  we decompose  $v_\mu = v_{<\lambda/4} + \sum_{\lambda/4 \leq \rho \leq \mu} v_\rho$  and apply (2.23) to the first term and (2.24) (which we prove next) to the remaining terms.

We turn to estimate (2.24). It suffices to prove the estimate for  $\mu = 1 \leq \lambda$ . We begin again with  $L = 1$ . As above it suffices to consider a fixed number  $k \in \mathbb{Z}^2$ , which we even may assume to be zero. The summation with respect to  $k$  poses no difficulties. The  $L^4$  Strichartz estimate implies  $\|u_{\Gamma_{\lambda,k}}^2\|_{L^2} \leq c\lambda \|u_{\Gamma_{\lambda,k}}\|_{U_{KP}^4}^2$ . By Hölder's inequality for sequences and orthogonality

$$\sum_k \|(u_{\Gamma_{\lambda,k}} u_{\Gamma_{\lambda,k}})_{\Gamma_{1,k,\lambda}}\|_{l^r(L^2)} \leq c\lambda^{(\frac{2}{r}-1)+1} \|u_{\Gamma_{\lambda,k}}\|_{l^p V_{KP}^2} \|u_{\Gamma_{\lambda,k}}\|_{l^q V_{KP}^2}.$$

The condition  $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$  suffices for that summation. This time there will be an important modification for large  $L$ . As above, if  $k \geq 2$ , by the bilinear estimate of Theorem 2.1, and its consequences for  $U_{KP}^2$ ,

$$(2.27) \quad \|u_{\Gamma_{\lambda,0}} v_{\Gamma_{\lambda,k,L}}\|_{L^2} \leq c\lambda L^{-1} \|u_{\Gamma_{\lambda,0}}\|_{U_{KP}^2} \|v_{\Gamma_{\lambda,k,L}}\|_{U_{KP}^2}.$$

As above we have to sum over  $L^2$  terms which gives

$$\|T_L(u_{\Gamma_{\lambda,k,L}}, v_{\Gamma_{\lambda,k',L}})\|_{L^2} \leq cL^{1-\frac{2}{p}+(1-\frac{2}{q})+\varepsilon} \lambda \|u_{\Gamma_{\lambda,k,L}}\|_{l^p V_{KP}^2} \|v_{\Gamma_{\lambda,k,L}}\|_{l^q V_{KP}^2}.$$

We complete the proof with the same type of approximation and summation as above.  $\square$

### 3. PROOF OF THE MAIN THEOREM

**3.1. A simple proof with three flaws.** We begin with sketching an incomplete proof, attempting to get an iteration argument work in a simpler and slightly larger space  $X^0$  defined by the norm

$$\|u\|_{X^0} = \sup_{\lambda > 0} \left( \lambda^{1/2} \|u_\lambda\|_{V_{KP}^2} + \lambda^{-1} \|u_\lambda\|_{\dot{X}^{0,1}} \right).$$

This will almost work, and we will provide essential modifications which will complete the wellposedness argument. Existence via the contraction mapping principle follows from the two estimates

$$(3.1) \quad \lambda^{\frac{1}{2}} \left\| \int_0^t S(t-s) \partial_x(uv)_\lambda ds \right\|_{V_{KP}^2} \leq c \|u\|_{X^0} \|v\|_{X^0}$$

and

$$(3.2) \quad \lambda^{-1} \left\| \int_0^t S(t-s) \partial_x(uv)_\lambda ds \right\|_{\dot{X}^{0,1}} \leq c \|u\|_{X^0} \|v\|_{X^0}.$$

It is useful to observe that

$$(3.3) \quad \lambda^{1/2} \|u_\lambda\|_{V_{KP}^2} + \lambda^{-1} \|u_\lambda\|_{\dot{X}^{0,1}} \sim \lambda^{1/2} \|u_\lambda^{\leq \lambda^3}\|_{V_{KP}^2} + \lambda^{-1} \|u_\lambda^{> \lambda^3}\|_{\dot{X}^{0,1}}.$$

This implies (3.3).

By scaling it suffices to consider (3.1) and (3.2) for  $\lambda = 1$ , and duality reduces the two estimates to bounds for trilinear integrals

$$(3.4) \quad \int uvw_1 dx dy dt = \int_S \widehat{u}(\xi_1, \eta_1, \tau_1) \widehat{v}(\xi_2, \eta_2, \tau_2) \widehat{w_1}(\xi_3, \eta_3, \tau_3) d\mathcal{H}^8.$$

for  $w_1 \in V_{KP}^2 \cup L^2$ . Here  $S$  denotes the subspace of dimension 8 given by

$$\{\xi_1 + \xi_2 + \xi_3 = 0, \eta_1 + \eta_2 + \eta_3 = 0, \tau_1 + \tau_2 + \tau_3 = 0\}$$

and  $d\mathcal{H}^8$  denotes the 8-dimensional Hausdorff measure on it. On this subspace (2.12) becomes

$$(3.5) \quad \tau_1 - \omega_1 + \tau_2 - \omega_2 + \tau_3 - \omega_3 = -3\xi_1\xi_2\xi_3 - \frac{\xi_1\xi_2}{\xi_3} \left| \frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2} \right|^2.$$

It has the following important interpretation: If  $\tau_i = \xi_i^3 - \eta_i^2/\xi_i$  for  $i = 1, 2$  then  $\Lambda \geq |\xi_1\xi_2(\xi_1 + \xi_2)|$  where

$$\begin{aligned} \Lambda := \left| (\tau_2 - \tau_1) - (\xi_2 - \xi_1)^3 + \frac{|\eta_2 - \eta_1|^2}{\xi_2 - \xi_1} \right| &= |\xi_1||\xi_2||\xi_1 + \xi_2| \\ &\quad + \frac{|\xi_1||\xi_2|}{|\xi_1 + \xi_2|} \left| \frac{\eta_1}{\xi_1} - \frac{\eta_2}{\xi_2} \right|^2 \end{aligned}$$

$\Lambda$  is a function of  $\xi_i$  and  $\eta_i$ . We decompose  $u, v$  into dyadic pieces according to the size of  $\xi$ 's and, by an abuse of notation we choose a version which is constant on the sets of consideration. We decompose  $u_i = u_i^{>\Lambda/3} + u_i^{\leq\Lambda/3}$ . Then the trilinear integral vanishes unless at least one term has high modulation since  $\int u_1^{\leq\Lambda/3} u_2^{\leq\Lambda/3} u_3^{\leq\Lambda/3} dx dy dt = 0$ . The Strichartz estimates give for  $\lambda \geq 1$

$$(3.6) \quad \begin{aligned} \int u_\lambda v_\lambda w_1 dx dy dt &\leq \|w_1\|_{L^2} \|u_\lambda v_\lambda\|_{L^2} \\ &\leq C \|w_1\|_{L^2} \left( \lambda^{\frac{1}{2}} \|u_\lambda\|_{V_{KP}^2} \right) \left( \lambda^{\frac{1}{2}} \|v_\lambda\|_{V_{KP}^2} \right) \end{aligned}$$

which yields by scaling and orthogonality of the Paley-Littlewood pieces

$$\left\| \partial_x \int_0^t S(t-s) u_\lambda v_\lambda ds \right\|_{\dot{X}^{0,1}} \leq C \left( \lambda^{\frac{1}{2}} \|u_\lambda\|_{V_{KP}^2} \right) \left( \lambda^{\frac{1}{2}} \|v_\lambda\|_{V_{KP}^2} \right).$$

By the bilinear estimate of Theorem 2.1 - see also (2.17)

$$(3.7) \quad \begin{aligned} \left| \int u_\lambda^{>\lambda^2/3} v_\lambda w_1 dx dy dt \right| &\leq \|u_\lambda^{>\lambda^2/3}\|_{L^2} \|v_\lambda w_1\|_{L^2} \\ &\leq c \lambda^{-1} \|u_\lambda^{>\lambda^3/3}\|_{V_{KP}^2} \|v_\lambda\|_{U_{KP}^2} \|w_1\|_{U_{KP}^2} \end{aligned}$$

and hence

$$\begin{aligned} \left\| \partial_x \int_0^t S(t-s) (u_\lambda v_\lambda)_1 ds \right\|_{\dot{X}^{0,1}} &+ \left\| \partial_x \int_0^t S(t-s) (u_\lambda v_\lambda)_1 ds \right\|_{V_{KP}^2} \\ &\leq c \lambda^{\frac{1}{2}} \|u_\lambda\|_{U_{KP}^2} \lambda^{\frac{1}{2}} \|v_\lambda\|_{U_{KP}^2}. \end{aligned}$$

For  $\mu \leq 1$  we estimate using the Strichartz estimate (2.15) for  $p = q = 4$  and the embedding  $V_{KP}^2 \subset U_{KP}^4$

$$(3.8) \quad \begin{aligned} \int u_\mu^{>\mu/3} v_1 w_1 dx dy dt &\leq \|u_\mu^{>\mu/3}\|_{L^2} \|v_1 w_1\|_{L^2} \\ &\leq c (\mu^{-1} \|u_\mu\|_{\dot{X}^{0,1}}) \|v_1\|_{V_{KP}^2} \|w_1\|_{V_{KP}^2} \end{aligned}$$



and the bilinear estimate (2.17) to arrive at

$$(3.9) \quad \begin{aligned} \int u_\mu v_1^{>\mu/3} w_1 dx dy dt &\leq c\mu \|u_\mu\|_{U_{KP}^2} \mu^{-\frac{1}{2}} \|v_1\|_{V_{KP}^2} \|w_1\|_{U_{KP}^2} \\ &= c(\mu^{1/2} \|u_\mu\|_{U_{KP}^2}) \|v_1\|_{V_{KP}^2} \|w_1\|_{U_{KP}^2}, \end{aligned}$$

thus

$$\begin{aligned} &\left\| \partial_x \int_0^t S(t-s)(u_\mu v_1)_1 ds \right\|_{\dot{X}^{0,1}} + \left\| \partial_x \int_0^t S(t-s)(u_\mu v_1)_1 ds \right\|_{V_{KP}^2} \\ &\leq c \left( \mu^{\frac{1}{2}} \|u_\mu\|_{U_{KP}^2} + \mu^{-1} \|u_\mu\|_{\dot{X}^{0,1}} \right) \|v_1\|_{U_{KP}^2}. \end{aligned}$$

To achieve (3.1) and (3.2), there are three issues to resolve:

- i) The summability with respect to  $\lambda$  and  $\mu$  requires improved estimates to obtain (3.1) and (3.2).
- ii) In (3.7) and (3.9), we have to replace  $U_{KP}^2$  by  $V_{KP}^2$ .
- iii) The function  $u = S(t)u_0$  for  $t > 0$  and  $u = 0$  for  $t < 0$  is not in  $\dot{X}^{0,1}$ . We need a variant of the estimates for solutions to the homogeneous initial value problem.

Here as always we oversimplify things a bit: We have to consider more general frequency combinations, and we only know that the two highest frequencies have to be of comparable size, otherwise the trilinear integral vanishes, which as always we ignore since we want to keep the formulas simpler, and there is no new difficulty connected with that.

**3.2.  $l^p$  summation and bilinear estimate.** We begin to explain the modifications for the proof. We use  $l^q l^p(V_{KP}^2)$  with  $1 \leq q \leq \infty, 1 < p < 2$  and replace  $\dot{X}^{0,1}$  by  $\dot{X}^{0,b}$  with some  $b \in (\frac{5}{6}, 1)$  as discussed in the introduction.

**Definition 3.1.** Let  $X$  be the space of all distributions for which

$$\|u\|_X := \left\| \lambda^{\frac{1}{2}} \|u_\lambda\|_{l^p V_{KP}^2} + \lambda^{2-3b} \|u_\lambda\|_{\dot{X}^{0,b}} \right\|_{l_\lambda^q} < \infty.$$

We next formulate a bilinear estimate.

**Proposition 3.2** (Bilinear estimates for the quadratic term). *For  $u, v \in X$ , we have*

$$(3.10) \quad \left\| \int_{-\infty}^t S(t-s) \partial_x(uv) ds \right\|_X \leq c \|u\|_X \|v\|_X.$$

In our proof we obtain a slightly stronger bilinear estimate. We will replace the  $U_{KP}^2$  by  $V_{KP}^2$  at several places.

*Proof.* Using a Littlewood-Paley decomposition, a duality argument and an expansion of (3.10) the estimate follows from the next four inequalities. The high  $\times$  high to low type estimates are

$$(3.11) \quad \begin{aligned} \int u_\lambda v_\lambda w_\mu dx dy dt &\leq C \mu^{3b-3} \left( \frac{\mu}{\lambda} \right)^{2-2b-\varepsilon} \\ &\times \left( \lambda^{\frac{1}{2}} \|u_\lambda\|_{l^p V_{KP}^2} \right) \left( \lambda^{\frac{1}{2}} \|v_\lambda\|_{l^p V_{KP}^2} \right) \|w_\mu\|_{\dot{X}^{0,1-b}} \end{aligned}$$

$$(3.12) \quad \int u_\lambda v_\lambda w_\mu dx dy dt \leq C \mu^{-\frac{3}{2}} \left( \frac{\mu}{\lambda} \right)^{2-\frac{2}{p}-\varepsilon} \\ \times \left( \lambda^{\frac{1}{2}} \|u_\lambda\|_{l^p V_{KP}^2} \right) \left( \lambda^{\frac{1}{2}} \|v_\lambda\|_{l^p V_{KP}^2} \right) \|w_\mu\|_{l^{p'} V_{KP}^2}.$$

which we complement by low  $\times$  high to high estimates

$$(3.13) \quad \int u_\mu v_\lambda w_\lambda dx dy dt \leq c \lambda^{-\frac{3}{2}} \left( \frac{\mu}{\lambda} \right)^{\min\{\frac{2}{p}-1-\varepsilon, 2b-\frac{5}{3}\}} \\ \times \left( \mu^{1/2} \|u_\mu\|_{l^p V_{KP}^2} + \mu^{2-3b} \|u_\mu\|_{\dot{X}^{0,b}} \right) \left( \lambda^{\frac{1}{2}} \|v_\lambda\|_{l^p V_{KP}^2} \right) \|w_\lambda\|_{l^{p'} V_{KP}^2}$$

$$(3.14) \quad \int u_\mu v_\lambda w_\lambda dx dy dt \leq c \lambda^{3b-3} \left( \frac{\mu}{\lambda} \right)^{\min\{b-\frac{1}{2}-\varepsilon, 3b-\frac{5}{2}\}} \\ \times \left( \mu^{1/2} \|u_\mu\|_{l^p V_{KP}^2} + \mu^{2-3b} \|u_\mu\|_{\dot{X}^{0,b}} \right) \left( \lambda^{\frac{1}{2}} \|v_\lambda\|_{l^p V_{KP}^2} \right) \|w_\lambda\|_{\dot{X}^{0,1-b}}$$

for  $\mu \leq \lambda$ . Proposition 3.2 and more precisely (3.10) follows by summing up the  $\mu$  and  $\lambda$ , which is trivial. More precisely we would have to consider frequencies  $\lambda_1$  and  $\lambda_2$  for the first estimates, but, since on the Fourier side the Fourier variables  $\xi_1$  and  $\xi_2$  have to add up to something of size  $\sim \mu$  which we assume always less than  $\lambda$ , it suffices to consider neighboring dyadic intervals resp  $\lambda_1 \sim \lambda_2$ . To simplify the notation we restrict to  $\lambda_1 = \lambda_2 = \lambda$  and we deal similarly with the other inequalities.

We turn to the proof of the four main estimates (3.11)-(3.14). For the [(high,high) $\rightarrow$  low] type estimates (3.11) and (3.12), by rescaling, we assume that  $\mu = 1$ . We decompose

$$\int u_\lambda v_\lambda w_1 dx dy dt = \sum_{L \in 2^{\mathbb{N}}} \int T_L(u_\lambda v_\lambda) w_1 dx dy dt$$

where the sum runs over  $L = 2^{\mathbb{Z}_+}$ .

At least one of the terms has to have high modulation, i.e. modulation at least  $\geq L^2 \lambda^2 / 3$ . For simplicity we will ignore the denominator 3. Now, if  $L > 1$  - the difference for  $L = 1$  is only in notation -

$$(3.15) \quad \left| \int T_L(u_\lambda, v_\lambda) w_1^{\geq L^2 \lambda^2} dx dy dt \right| \leq \|T_L(u_\lambda, v_\lambda)_1\|_{l^2 L^2} \|w_1^{\geq L^2 \lambda^2}\|_{l^2 L^2} \\ \leq C \lambda \|u_\lambda\|_{l^2 V_{KP}^2} \|v_\lambda\|_{l^2 V_{KP}^2} (L\lambda)^{-2(1-b)} \|w_1\|_{l^2 \dot{X}^{0,1-b}}.$$

Since for  $1 < p < 2$ ,

$$\|w_1\|_{l^2 \dot{X}^{0,1-b}} \approx \|w_1\|_{\dot{X}^{0,1-b}}, \|u_\lambda\|_{l^2 V_{KP}^2} \leq \|u_\lambda\|_{l^p V_{KP}^2}$$

we obtain

$$\sum_L \left| \int T_L(u_\lambda v_\lambda) w_1^{\geq L^2 \lambda^2} dx dy dt \right| \\ \leq c \lambda^{2b-2} \left( \lambda^{1/2} \|u_\lambda\|_{l^p V_{KP}^2} \right) \left( \lambda^{1/2} \|v_\lambda\|_{l^p V_{KP}^2} \right) \|w_1\|_{\dot{X}^{0,1-b}}.$$

(3.15) can also be bounded, for  $1 < p < 2$ , by

$$\lambda (L\lambda)^{\frac{2}{p}-1} \|u_\lambda\|_{l^p V_{KP}^2} \|v_\lambda\|_{l^p V_{KP}^2} (L\lambda)^{-1} \|w_1\|_{l^{p'} V_{KP}^2}.$$

Here we used Hölder's inequality and then the high modulation estimate for  $w$  and (2.24) with  $r = q = p$  for the product. We complete the proof of (3.11) for the case

the  $w$  has high modulation by

$$\begin{aligned} & \sum_L \left| \int T_L(u_\lambda v_\lambda) w_1^{\geq L^2 \lambda^2} dx dy dt \right| \\ & \leq c \lambda^{\frac{2}{p}-2} \left( \lambda^{1/2} \|u_\lambda\|_{l^p V_{KP}^2} \right) \left( \lambda^{1/2} \|v_\lambda\|_{l^p V_{KP}^2} \right) \|w_1\|_{l^{p'} V_{KP}^2}. \end{aligned}$$

Next we use the symmetry property of Lemma 2.4 to deal with the case that  $v$  has high modulation:

$$\begin{aligned} (3.16) \quad & \left| \int T_L(u_\lambda, v_\lambda^{\geq L^2 \lambda^2})_1 w_1^{< L^2 \lambda^2} dx dy dt \right| = \left| \int v_\lambda^{\geq L^2 \lambda^2} T_L(u_\lambda, w_1^{L^2 \lambda^2})_\lambda dx dy dt \right| \\ & \leq \|T_L(u_\lambda, w_1^{< L^2 \lambda^2})_\lambda\|_{l^2 L^2} \|v_\lambda^{\geq L^2 \lambda^2}\|_{l^2 L^2} \\ & \leq C \lambda^{\frac{2}{p}-1+\varepsilon} L^\varepsilon \|u_\lambda\|_{l^p V_{KP}^2} \|w_1\|_{l^{p'} V_{KP}^2} (L\lambda)^{-1} \|v_\lambda\|_{l^p V_{KP}^2}. \end{aligned}$$

with the obvious modification if  $L = 1$ . Here we used the high modulation estimate for  $v_\lambda$  and (2.23) with  $r = 2$ , and  $q = p'$ . The summation with respect to  $L$  gives

$$\begin{aligned} & \sum_L \left| \int T_L(u_\lambda v_\lambda^{\geq L^2 \lambda^2}) w_1^{< L^2 \lambda^2} dx dy dt \right| \\ & \leq C \lambda^{\frac{2}{p}-3+\varepsilon} \left( \lambda^{1/2} \|u_\lambda\|_{l^p V_{KP}^2} \right) \|w_1\|_{l^{p'} V_{KP}^2} \left( \lambda^{1/2} \|v_\lambda\|_{l^p V_{KP}^2} \right). \end{aligned}$$

In the same way, we can bound (3.16) by

$$\lambda^\varepsilon \|u_\lambda\|_{l^2 V_{KP}^2} \|w_1^{< L^2 \lambda^2}\|_{l^2 V_{KP}^2} (L\lambda)^{-1} \|v_\lambda\|_{l^2 V_{KP}^2}.$$

Notice that

$$\|f^{\leq L^2 \lambda^2}\|_{V_{KP}^2} \lesssim L^{2b-1} \lambda^{2b-1} \|f\|_{\dot{X}^{0,1-b}}.$$

(3.11) and (3.12) follows by a trivial summation over  $L$ .

Now we turn to (3.13) and (3.14) and rescale to  $\lambda = 1$ . We decompose the factors in the same fashion as above

$$\int u_\mu v_1 w_1 dx dy dt = \sum_L \int T_L(u_\mu v_1) w_1 dx dy dt.$$

As above, using (2.23) with  $r = q = p = 2$

$$\begin{aligned} & \left| \int T_L(u_\mu v_1) w_1^{\geq \mu L^2} dx dy dt \right| \leq \|T_L(u_\mu v_1)_1\|_{l^2 L^2} \|w_1^{\geq \mu L^2}\|_{l^2 L^2} \\ & \leq C \mu^{\frac{1}{2}} (L/\mu)^\varepsilon (\mu L^2)^{b-1} (\mu^{1/2} \|u_\mu\|_{l^2 V_{KP}^2}) \|v_1\|_{l^2 V_{KP}^2} \|w_1\|_{l^2 \dot{X}^{0,1-b}} \end{aligned}$$

resp. taking  $r = p = q < 2$ ,

$$\begin{aligned} & \left| \int T_L(u_\mu v_1) w_1^{\geq \mu L^2} dx dy dt \right| \leq \|T_L(u_\mu v_1)_1\|_{l^p L^2} \|w_1^{\geq \mu L^2}\|_{l^{p'} L^2} \\ & \leq C (L/\mu)^{1-\frac{2}{p}+\varepsilon} L^{-1} \mu^{1/2} \|u_\mu\|_{l^p V_{KP}^2} \|v_1\|_{l^p V_{KP}^2} \|w_1\|_{l^{p'} V_{KP}^2}. \end{aligned}$$

The summation with respect to  $L$  gives

$$\begin{aligned} & \sum_L \left| \int T_L(u_\mu v_1) w_1^{\geq \mu L^2} dx dy dt \right| \leq C \mu^{b-\frac{1}{2}-\varepsilon} \\ & \quad \times \left( \mu^{1/2} \|u_\mu\|_{l^p V_{KP}^2} \right) \|v_1\|_{l^p V_{KP}^2} \|w_1\|_{\dot{X}^{0,1-b}}. \end{aligned}$$

resp.

$$\begin{aligned} \sum_L \left| \int T_L(u_\mu v_1) w_1^{\geq \mu L^2} dx dy dt \right| &\leq C \mu^{\frac{2}{p}-1-\varepsilon} \\ &\times \left( \mu^{1/2} \|u_\mu\|_{l^p V_{KP}^2} \right) \|v_1\|_{l^p V_{KP}^2} \|w_1\|_{l^{p'} V_{KP}^2}. \end{aligned}$$

The same computation gives

$$\begin{aligned} \sum_L \left| \int T_L(u_\mu w_1) v_1^{\geq \mu L^2} dx dy dt \right| &\leq C \mu^{\frac{2}{p}-\frac{1}{2}-\varepsilon} \\ &\times \left( \mu^{1/2} \|u_\mu\|_{l^p V_{KP}^2} \right) \|v_1\|_{l^p V_{KP}^2} \|w_1\|_{l^{p'} V_{KP}^2} \end{aligned}$$

resp.

$$\begin{aligned} \sum_L \left| \int T_L(u_\mu w_1^{\leq \mu L^2}) v_1^{\geq \mu L^2} dx dy dt \right| &\leq C \mu^{b-\frac{1}{2}-\varepsilon} \\ &\times \left( \mu^{1/2} \|u_\mu\|_{l^2 V_{KP}^2} \right) \|v_1\|_{l^2 V_{KP}^2} \|w_1\|_{\dot{X}^{0,1-b}}. \end{aligned}$$

Here we used

$$\|w_1^{\leq \mu L^2}\|_{V_{KP}^2} \leq C \mu^{b-\frac{1}{2}} L^{2b-1} \|w_1\|_{\dot{X}^{0,1-b}}.$$

The last term with the high modulation on  $u_\mu$  is different, and it is the most interesting:

$$\left| \int u_\mu^{\geq \mu L^2} T_L(v_1, w_1) dx dy dt \right| \leq \|T_L(v_1, w_1)_\mu\|_{l^2 L_t^2 L_{xy}^{3/2}} \|u_\mu^{\geq \mu L^2}\|_{l^2 L_t^2 L_{xy}^3}.$$

We continue with the endpoint Strichartz estimate

$$\|u_\mu^{\geq \mu L^2}\|_{L_t^2 L^3} \leq \|u_\mu^{\geq \mu L^2}\|_{L^2}^{1/2} \|u_\mu^{\geq \mu L^2}\|_{L_t^2 L^6}^{1/2} \leq C(\mu L^2)^{\frac{1}{4}-b} \mu^{\frac{1}{12}} \|u_\mu\|_{\dot{X}^{0,b}}$$

for each part localized in  $\eta$  and we achieve

$$\begin{aligned} &\left| \int u_\mu^{\geq \mu L^2} T_L(v_1 w_1) dx dy dt \right| \\ &\leq C \mu^{2b-\frac{5}{3}} L^{\frac{1}{2}-2b+\frac{2}{p}-1} \|v_1\|_{l^p L_t^4 L^3} \|w_1\|_{l^{p'} L_t^4 L^3} \mu^{2-3b} \|u_\mu\|_{\dot{X}^{0,b}}. \end{aligned}$$

By Proposition 2.5, we drop  $T_L$  here. The exponent  $(4, 3)$  is a Strichartz pair. The summation with respect to  $L$  is trivial. It gives

$$\begin{aligned} \sum_L \left| \int u_\mu^{\geq \mu L^2} T_L(v_1, w_1) dx dy dt \right| \\ \leq c \mu^{2b-\frac{5}{3}} \left( \mu^{2-3b} \|u_\mu\|_{l^p \dot{X}^{0,b}} \right) \|v_1\|_{l^p V_{KP}^2} \|w_1\|_{l^{p'} V_{KP}^2}. \end{aligned}$$

resp.

$$\begin{aligned} \sum_L \left| \int u_\mu^{\geq \mu L^2} T_L(v_1, w_1^{\leq \mu L^2}) dx dy dt \right| \\ \leq c \mu^{3b-\frac{5}{2}} \left( \mu^{2-3b} \|u_\mu\|_{\dot{X}^{0,b}} \right) \|v_1\|_{V_{KP}^2} \|w_1\|_{\dot{X}^{0,1-b}}. \end{aligned}$$

The summation with respect to  $\mu$  requires  $b > \frac{5}{6}$  and we arrive at (3.13) and (3.14)

□

**3.3. The initial data, the proof of wellposedness.** It remains to consider estimate  $S(t)u_0$  in terms of the initial data. Let

$$\tilde{u}(t) = \chi_{[0,\infty)} S(t)u_0.$$

As we pointed out in issue iii), it is not in  $\dot{X}^{0,b}$  for any  $1/2 < b \leq 1$ , thus it is not in  $X$  unless it is trivial. Let

$$\|u\|_Y = \left\| \lambda^{1/2} \|u_\lambda\|_{l^p U_{KP}^1} \right\|_{l_\lambda^q}$$

to shorten the notation. Then by construction

$$\|\tilde{u}\|_Y \leq \|u_0\|_{l^q l^p L^2}.$$

The two estimates of the following proposition will allow to complete the proof.

**Proposition 3.3.** *The following estimates hold.*

$$(3.17) \quad \left\| \int_0^t S(t-s) \partial_x(uv) \right\|_X \leq c \|u\|_Y \|v\|_Y,$$

$$(3.18) \quad \left\| \int_0^t S(t-s) \partial_x(uv) \right\|_X \leq c \|u\|_X \|v\|_Y.$$

With these estimates at hand we complete the fixed point argument. By Duhamel's formula, to solve (1.1) on  $[0, \infty)$  is equivalent to solving

$$w = \tilde{u} + \int_0^t S(t-s) \partial_x(w^2)(s) ds.$$

We rewrite this equation in terms of the difference  $u = w - \tilde{u}$  and define the map

$$(3.19) \quad \begin{aligned} \Phi(u) &:= \int_0^t S(t-s) \partial_x((u + \tilde{u})^2)(s) ds \\ &= \int_0^t S(t-s) \partial_x \tilde{u}^2 ds + \int_0^t S(t-s) \partial_x(2\tilde{u}u + u^2) ds \end{aligned}$$

where we set  $u(s) = 0$  for  $s < 0$ .

Set  $r := \min(\frac{1}{4C}, 3\varepsilon)$ . Here  $C$  is the largest constant among the constants from (3.10), (3.17) and (3.18). We define the closed ball of radius  $r$  in  $X$

$$B_r := \{u \in X; \|u\|_X \leq r\}.$$

We search an unique fixed point of  $\Phi$  in  $B_r$ . By the definition of  $Y$

$$\|\tilde{u}\|_Y \leq C \|u\|_{l^q l^p L^2} \lesssim \varepsilon.$$

By (3.10), (3.17) and (3.18), we have

$$(3.20) \quad \|\Phi(u)\|_X \lesssim \|u\|_X^2 + 2\|\tilde{u}\|_Y \|u\|_X + \|\tilde{u}\|_Y^2 \leq r$$

and

$$\begin{aligned} \|\Phi(u) - \Phi(v)\|_X &\leq C \|u - v\|_X (\|u\|_X + \|v\|_X + \|\tilde{u}\|_Y) \\ &\leq \frac{1}{2} \|u - v\|_X. \end{aligned}$$

We apply the contraction mapping theorem to obtain existence of a unique fixed point. The linearization at the fixed point is invertible - it is a contraction by

construction - and the map  $\Phi$  is analytic. Hence the map from the initial data to the fixed point is analytic.

The estimate

$$\|u\|_X \leq C \|u_0\|_{l^q l^p L^2}^2$$

follows from (3.20). This completes the proof, up to proving Proposition 3.3.

**3.4. The proof of Proposition 3.3.** By the same strategy as above we continue to assume  $\mu \leq 1 \leq \lambda$ . The estimates (3.11) and (3.12) are in terms of  $l^p V_{KP}^2$  at frequency  $\lambda$ . It is a consequence of Minkowski's inequality that

$$\|u_\lambda\|_{l^p V_{KP}^2} \lesssim \|u_\lambda\|_{l^p U_{KP}^2} \lesssim \|u_\lambda\|_{l^p U_{KP}^1}.$$

We can directly replace  $l^p V_{KP}^2$  by  $l^p U_{KP}^1$  in the estimates (3.11) and (3.12). This completes the argument for the [(high,high)  $\rightarrow$  low] case, for both estimates (3.17) and (3.18). The next lemma provides the remaining [(low,high)  $\rightarrow$  high] estimates.

**Lemma 3.4.** *The following estimates hold, for  $\mu \leq 1$ ,*

$$(3.21) \quad \left\| \int_0^t S(t-s)(u_\mu v_1)_1 ds \right\|_{l^p(U_{KP}^2)} \leq C \mu^{\min(\frac{2}{p}-1-\varepsilon, b-\frac{1}{2})} \left( \mu^{1/2} \|u_\mu\|_{l^p U_{KP}^1} \right) \|v_1\|_{l^p U_{KP}^1},$$

$$(3.22) \quad \left\| \int_0^t S(t-s)(u_\mu v_1)_1 \right\|_{\dot{X}^{0,b}} \leq C \mu^{\min(\frac{2}{p}-1-\varepsilon, b-\frac{1}{2})} \left( \mu^{\frac{1}{2}} \|u_\mu\|_{l^p U_{KP}^1} \right) \|v_1\|_{l^p V_{KP}^2}.$$

$$(3.23) \quad \left\| \int_0^t S(t-s)(u_\mu v_1)_1 \right\|_{\dot{X}^{0,b}} \leq C \mu^{\min(\frac{2}{p}-1-\varepsilon, b-\frac{1}{2})} \left( \mu^{\frac{1}{2}} \|u_\mu\|_{l^p V_{KP}^2} \right) \|v_1\|_{l^p U_{KP}^1}.$$

Together with the versions of (3.11) and (3.12) above these imply (3.18) then (3.17) in Proposition 3.3 by an easy summation.

*Proof.* Again we use duality and decompose

$$\left| \int u_\mu v_1 w_1 dx dy dt \right| \leq \sum_L \left| \int u_\mu T_L(v_1, w_1) dx dy dt \right|.$$

At least one term has modulation  $\geq \mu L^2$ . Notice that

$$\|u_\lambda\|_{l^p(V_{KP}^2)} \lesssim \|u_\lambda\|_{l^p U_{KP}^1},$$

the estimates in (3.13) and (3.14) work well except the case  $u_\mu$  has the high modulation

$$\int u_\mu^{>\mu L^2} T_L(v_1 w_1) dx dy dt.$$

Let  $L \geq 1$  and consider

$$\int u_{\Gamma_{\mu,k,L/\mu}}^{>\mu L^2} T_L(v_{\Gamma_{1,k',L}}, u_{\Gamma_{1,k,L}}) dx dy dt$$

with  $16 \leq |k - k'| \leq 1000$  if  $L > 1$ , resp  $|k - k'| \leq 200$  if  $L = 1$ . we decompose  $u_{\Gamma_{\mu, k, \frac{L}{\mu}}}$  further

$$\begin{aligned}
& \sum_{16 \leq |k - k'| \leq 1000} \sum_{|k - l| \lesssim \frac{L}{\mu}} \left| \int u_{\Gamma_{\mu, l}}^{>\mu L^2} T_L(v_{\Gamma_{1, k, L}}, w_{\Gamma_{1, k, L}}) dx dy dt \right| \\
& \lesssim \sum_{16 \leq |k - k'| \leq 1000} \sum_{|k - l| \lesssim \frac{L}{\mu}} \|u_{\Gamma_{\mu, l}}^{>\mu L^2}\|_{L_t^1 L^\infty} \|T_L(v_{\Gamma_{1, k', L}}, w_{\Gamma_{1, k, L}})\|_{L_t^\infty L^1} \\
& \lesssim \sup_k \mu^{\frac{5}{2}} \left(\frac{L}{\mu}\right)^{\frac{2}{p'}} \|u_{\Gamma_{\mu, k, \frac{L}{\mu}}}\|_{l^p(L_t^1 L^2)} \|v_1\|_{l^p L_t^\infty L^2} L^{\frac{2}{p}-1} \|w_1\|_{l^{p'} L_t^\infty L^2} \\
& \lesssim \mu^{\frac{2}{p}-1} L^{-1} \left(\mu^{\frac{1}{2}} \|u_\mu\|_{l^p U_{KP}^1}\right) \|v_1\|_{l^p V_{KP}^2} \|w_1\|_{l^{p'} V_{KP}^2}.
\end{aligned}$$

Here we used the size of the set  $\Gamma_{\mu, l}$  is  $\mu^5$ . We estimate similarly to above

$$\begin{aligned}
& \sum_{16 \leq |k - k'| \leq 1000} \left| \int u_{\Gamma_{\mu, k, L/\mu}}^{>\mu L^2} T_L(v_{\Gamma_{1, k, L}}, w_{\Gamma_{1, k, L}}) dx dy dt \right| \\
& \lesssim \mu^b L^{2b-2} \|u_\mu\|_{l^2 U_{KP}^1} \|v_1\|_{l^2 V_{KP}^2} \|w_1\|_{\dot{X}^{0, 1-b}}.
\end{aligned}$$

Here we applied Sobolev's resp. Bernstein's inequality in sets of Fourier size  $\mu^3 L^2$  and the high modulation factor  $\mu L^2$ . The summation with respect to  $L$  is trivial since the exponent is negative. Finally (3.23) is a direct consequence of (3.14).  $\square$

#### 4. ILL-POSEDNESS AND FUNCTION SPACES

**4.1. Ill-posedness in  $l^q l^p L^2$  for  $p > 2$ .** We prove illposedness (Theorem 4) by contradiction. By scaling it suffices to consider  $T = 1$ . Suppose that the flow map  $u_0 \rightarrow u(1)$  defines a map from  $l^q l^p L^2$  to itself which is continuously differentiable near 0, and twice differentiable at 0, for some  $p > 2$ . For simplicity we choose  $q = \infty$ , but the proof works for all  $q \in [1, \infty]$ .

Consider the Cauchy problem

$$(4.1) \quad \begin{cases} \partial_x (\partial_t u + \partial_x^3 u + \partial_x(u^2)) + \triangle_y u = 0 \\ u(0, x, y) = \gamma \phi(x, y) \quad \gamma \in \mathbb{R}. \end{cases}$$

where  $\phi \in l^\infty l^p L^2$  and  $1 < p < \infty$ . Suppose that  $u(\gamma, t, x, y)$  solves (4.1). By Duhamel's formula, we have

$$u(\gamma, t, x, y) = \gamma S(t) \phi(x, y) + \int_0^t S(t-s) \partial_x (u(\gamma, x, y)^2)(s) ds.$$

Since the flow map is (twice) differentiable at  $u_0 = 0$

$$\frac{\partial u}{\partial \gamma}(0, t, x, y) = S(t) \phi(x, y) := u_1(t, x, y),$$

$$\frac{\partial^2 u}{\partial \gamma^2}(0, t, x, y) = -2 \int_0^t S(t-s) \partial_x (u_1^2(s)) ds := u_2(t, x, y).$$

Since we assume the flow map to be twice differentiable

$$(4.2) \quad \|u_2(1, \cdot)\|_{l^\infty l^p L^2} \lesssim \|\phi\|_{l^\infty l^p L^2}^2.$$

We construct a sequence of initial data for  $u_1$  of norm 1 so that the norm of  $u_2(1)$  tends to infinity. This yields the desired contradiction.

We define the initial data  $\phi$  defined by its Fourier transform

$$\begin{aligned}\hat{\phi}(\xi, \eta) &= \frac{1}{\mu^3 \left(\frac{\lambda}{\mu}\right)^{\frac{2}{p}}} \chi_{[\frac{\mu}{2}, \mu]}(\xi) \chi_{[\frac{\lambda\mu}{2}, 2\lambda\mu]^2}(\eta) + \frac{1}{\mu^{\frac{3}{2}} \lambda^{\frac{3}{2}}} \chi_{[\lambda+\frac{\mu}{2}, \lambda+\mu]}(\xi) \chi_{[\frac{\lambda\mu}{2}, 2\lambda\mu]^2}(\eta) \\ &:= \hat{\phi}_1 + \hat{\phi}_2.\end{aligned}$$

Here dyadic numbers  $\mu \ll 1 \ll \lambda$  will be chosen later. It is easy to check that

$$\|\phi\|_{l^\infty l^p L^2} \approx \|\phi_1\|_{l^\infty l^p L^2} \approx \|\phi_2\|_{l^\infty l^p L^2} \approx 1.$$

Moreover

$$\begin{aligned}u_1 &= S(t)\phi_1 + S(t)\phi_2, \\ u_1^2 &= (S(t)\phi_1)^2 + (S(t)\phi_2)^2 + 2(S(t)\phi_1 S(t)\phi_2) := f_1 + f_2 + f_3.\end{aligned}$$

The Fourier transforms of the three summand are supported on pairwise disjoint sets and they are orthogonal. We then decompose  $u_2$  into three orthogonal parts as

$$u_2(1) = \int_0^1 S(1-s)(f_1 + f_2 + f_3)(s) ds := F_1 + F_2 + F_3.$$

By (4.2), we have

$$(4.3) \quad \|F_3(1)\|_{l^\infty l^p L^2} \lesssim \|u_2(1, \cdot)\|_{l^\infty l^p L^2} \lesssim 1.$$

By Lemma 4 in Page 376 of [12], we have

$$\hat{F}_3(1, \xi, \eta) = 2 \frac{\xi e^{i\omega(\xi, \eta)}}{\mu^3 \left(\frac{\lambda}{\mu}\right)^{\frac{2}{p}} \mu^{\frac{3}{2}} \lambda^{\frac{3}{2}}} \int_A \frac{e^{iR(\xi, \xi_1, \eta, \eta_1)} - 1}{R(\xi, \xi_1, \eta, \eta_1)} d\xi_1 d\eta_1.$$

Here  $R(\xi, \xi_1, \eta, \eta_1)$  denotes the resonance function

$$(4.4) \quad -3\xi\xi_1(\xi - \xi_1) - \frac{\xi\xi_1}{\xi - \xi_1} \left| \frac{\eta}{\xi} - \frac{\eta_1}{\xi_1} \right|^2.$$

In the set

$$\begin{aligned}A &= \left\{ \xi_1, \eta_1 : \xi_1 \in \left[\frac{\mu}{2}, \mu\right], \eta_1 \in \left[\frac{\lambda\mu}{2}, 2\lambda\mu\right]^2, \right. \\ &\quad \left. \xi - \xi_1 \in \left[\lambda + \frac{\mu}{2}, \lambda + \mu\right], \eta - \eta_1 \in \left[\frac{\lambda\mu}{2}, 2\lambda\mu\right]^2 \right\}\end{aligned}$$

the resonance function is bounded from below:

$$|R(\xi, \xi_1, \eta, \eta_1)| \sim \lambda^2 \mu.$$

If  $\mu\lambda^2 = O(1)$  (we may choose  $\mu$  and  $\lambda$ ) and obtain

$$\frac{e^{iR(\xi, \xi_1, \eta, \eta_1)} - 1}{R(\xi, \xi_1, \eta, \eta_1)} = 1 + O(1).$$

It follows that

$$|\hat{F}_3(1, \xi, \eta)| \geq \frac{\lambda\lambda^2\mu^3}{\mu^3 \left(\frac{\lambda}{\mu}\right)^{\frac{2}{p}} \mu^{\frac{3}{2}} \lambda^{\frac{3}{2}}} \chi_{[\lambda+\mu, \lambda+\frac{3\mu}{2}]}(\xi) \chi_{[\lambda\mu, 2\lambda\mu]^2}(\eta).$$

Then

$$(4.5) \quad 1 \gtrsim \|F_3\|_{l^\infty l^p L^2} \gtrsim \frac{\lambda^3 \mu^3}{\mu^3 \left(\frac{\lambda}{\mu}\right)^{\frac{2}{p}}} \approx \frac{\lambda^3}{\lambda^{\frac{6}{p}}}.$$



Here we used  $\mu\lambda^2 = O(1)$ . Since  $\lambda \gg 1$  we arrive at a contradiction to (4.5) unless  $p \leq 2$ .

**4.2. The function spaces  $l^q l^p L^2$ .** We prove Theorem 1.4. By the embedding  $l^q l^p L^2 \subset l^{\tilde{q}} l^{\tilde{p}} L^2$  if  $\tilde{q} \geq q$  and  $\tilde{p} \geq p$  it suffices to prove endpoint statements.

(i) Let  $f$  be a Schwartz function and fix  $\lambda$ . Trivially

$$\left( \sum_{l \in \lambda \cdot \mathbb{Z}^2} \|f_{\Gamma_{\lambda,l}}\|_{L^2}^p \right)^{\frac{1}{p}} = \left( \sum_{M \geq \lambda^2} \sum_{|l| \sim \frac{M}{\lambda}} \|f_{\Gamma_{\lambda,l}}\|_{L^2}^p \right)^{\frac{1}{p}}$$

and for  $\frac{4}{3} < p$  and  $N > 2$ , we have

$$\|f_{\Gamma_{\lambda,l}}\|_{L^2} \lesssim_N \frac{\lambda^{\frac{3}{2}}}{(1 + \lambda + M)^N},$$

thus

$$(4.6) \quad \left( \sum_{l \in \lambda \cdot \mathbb{Z}^2} \|f_{\lambda, \Gamma_{l,\lambda}}\|_{L^2}^p \right)^{\frac{1}{p}} \lesssim_N \frac{\lambda^{\frac{3}{2} - \frac{2}{p}}}{(1 + \lambda)^N}$$

and for  $p > 2$

$$\sum_{\lambda} \lambda^{-\frac{1}{2}} \|f_{\lambda}\|_{l^p(L^2)} < \infty.$$

By duality  $l^{\infty} l^p L^2$  ( $p < 2$ ) embeds into the space of distributions. A small modification shows that  $l^1 l^2 L^2$  embeds into the space of distributions.

(ii) It suffices to construct a sequence of Schwartz functions which converges in  $l^p l^2 L^2$  ( $p > 1$ ) but diverges as distributions. Since

$$l^p l^2 L^2 = L^2(\mathbb{R}^2; \dot{B}_{2,p}^{\frac{1}{2}})$$

it suffices to construct a sequence of functions  $\phi_{\mu}$  of one variable of norm 1 in  $\dot{B}_{2,q}^{\frac{1}{2}}$  and a Schwartz function  $\phi$  so that  $\int \phi_{\mu} \phi dx \rightarrow \infty$ . Here  $\dot{B}_{2,q}^{\frac{1}{2}}$  denotes the homogeneous Besov space. This is well known but we give an example for completeness. For  $0 < \lambda$  we choose a Schwartz function  $f_{\lambda}$  with the property

$$\hat{f}_{\lambda}(\xi) = \begin{cases} \lambda^{-1}, & \text{for } |\xi| \sim \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

For any fixed  $\mu \ll 1$ , we define

$$\phi_{\mu} = \frac{1}{|\ln \mu|^{\frac{1}{p}}} \sum_{\mu^2 \leq \lambda \leq \mu} f_{\lambda}.$$

It is easy to see

$$\|\phi_{\mu}\|_{\dot{B}_{2,q}^{\frac{1}{2}}} \sim 1.$$

However if  $\psi$  is a Schwartz function with Fourier transform supported in the ball  $B(0, 2)$  and  $\hat{\psi} = 1$  in the unit ball  $B(0, 1)$  then

$$\langle \psi, \phi_{\mu} \rangle \sim \sum_{\mu^2 \leq \lambda \leq \mu} |\ln \lambda|^{-\frac{1}{p}} \sim |\ln \mu|^{1 - \frac{1}{p}}.$$

(iii) Suppose now that the Schwarz function  $\phi$  is in  $l^\infty l^p L^2$  for  $p < \frac{4}{3}$ . We assume there exists  $(0, \eta_0) \in \mathbb{R}^3$  such that  $\hat{\phi}(0, \eta_0) \neq 0$ . By continuity, there exists  $r, c > 0$  such that

$$|\hat{\phi}(\xi, \eta)| > c, \text{ for } (\xi, \eta) \in B := B((0, \eta_0), r).$$

Then

$$\begin{aligned} \sup_{\lambda} \lambda^{\frac{1}{2}} \left( \sum_{l \in \lambda \cdot \mathbb{Z}^2} \|\phi_{\Gamma_{\lambda, l}}\|_{L^2}^p \right)^{\frac{1}{p}} &\geq \sup_{\lambda \lesssim r} \lambda^{\frac{1}{2}} \left( \sum_l \|\phi_{\Gamma_{\lambda, l} \cap B}\|_{L^2}^p \right)^{\frac{1}{p}} \\ &\sim \sup_{\lambda \lesssim r} r^{\frac{2}{p}} \lambda^{1 - \frac{2}{p}} \|\phi_{\lambda \cap B}\|_{L^2} \sim \sup_{\lambda \lesssim r} c r^{\frac{2}{p} + 1} \lambda^{\frac{3}{2} - \frac{2}{p}}, \end{aligned}$$

which is  $\infty$  if  $1 < p < \frac{4}{3}$ . This is a contradiction and hence

$$0 = \hat{\phi}(0, \eta) = (2\pi)^{-\frac{3}{2}} \int e^{-iy\eta} \phi(x, y) dx dy$$

for all  $\eta \in \mathbb{R}^2$ . The conclusion for  $l^q l^{\frac{4}{3}} L^2$  follows in the same fashion.

(iv) It follows from (4.6) that Schwartz functions are contained in  $l^\infty l^p L^2$  if  $\frac{4}{3} \leq p$  and in  $l^q l^p L^2$  if  $\frac{4}{3} < p$  and  $1 \leq q < \infty$ .

## REFERENCES

- [1] Jean Bourgain. On the Cauchy problem for the Kadomtsev-Petviashvili equation. *Geom. Funct. Anal.*, 3(4):315–341, 1993.
- [2] Martin Hadac. *On the local well-posedness of the Kadomtsev-Petviashvili II equation*. PhD thesis, Universität Dortmund, 2007.
- [3] Martin Hadac. Well-posedness for the Kadomtsev-Petviashvili II equation and generalisations. *Trans. Amer. Math. Soc.*, 360(12):6555–6572, 2008.
- [4] Martin Hadac, Sebastian Herr, and Herbert Koch. Well-posedness and scattering for the KP-II equation in a critical space. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(3):917–941, 2009.
- [5] Martin Hadac, Sebastian Herr, and Herbert Koch. Erratum to “Well-posedness and scattering for the KP-II equation in a critical space” [Ann. I. H. Poincaré—AN 26 (3) (2009) 917–941]. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27(3):971–972, 2010.
- [6] Pedro Isaza, Juan López, and JorgeJorge Mejía. The Cauchy problem for the Kadomtsev-Petviashvili (KP-II) equation in three space dimensions. *Comm. Partial Differential Equations*, 32(4-6):611–641, 2007.
- [7] Pedro Isaza and Jorge Mejía. Local and global Cauchy problems for the Kadomtsev-Petviashvili (KP-II) equation in Sobolev spaces of negative indices. *Comm. Partial Differential Equations*, 26(5-6):1027–1054, 2001.
- [8] Markus Keel and Terence Tao. Endpoint Strichartz estimates. *Amer. J. Math.*, 120(5):955–980, 1998.
- [9] Christian Klein and Jean-Claude Saut. Numerical study of blow up and stability of solutions of generalized Kadomtsev-Petviashvili equations. *J. Nonlinear Sci.*, 22(5):763–811, 2012.
- [10] Herbert Koch and Daniel Tataru. Dispersive estimates for principally normal pseudodifferential operators. *Comm. Pure Appl. Math.*, 58(2):217–284, 2005.
- [11] Herbert Koch, Daniel Tataru, and Monica Vişan. Dispersive equations and nonlinear waves. 2014.
- [12] Luc Molinet, Jean-Claude Saut, and Nickolay Tzvetkov. Well-posedness and ill-posedness results for the Kadomtsev-Petviashvili-I equation. *Duke Math. J.*, 115(2):353–384, 2002.
- [13] Hideo Takaoka. Well-posedness for the Kadomtsev-Petviashvili II equation. *Adv. Differential Equations*, 5(10-12):1421–1443, 2000.
- [14] Hideo Takaoka and Nickolay Tzvetkov. On the local regularity of the Kadomtsev-Petviashvili-II equation. *Internat. Math. Res. Notices*, (2):77–114, 2001.
- [15] Nickolay Tzvetkov. On the Cauchy problem for Kadomtsev-Petviashvili equation. *Comm. Partial Differential Equations*, 24(7-8):1367–1397, 1999.

- [16] Norbert Wiener. The quadratic variation of a function and its fourier coefficients. In Pesi Rustom Masani, editor, *Collected works with commentaries. Volume II: Generalized harmonic analysis and Tauberian theory; classical harmonic and complex analysis*, volume 15 of *Mathematicians of our Time*, chapter XIII, page 969. Cambridge, Mass, London: The MIT Press., 1979.

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